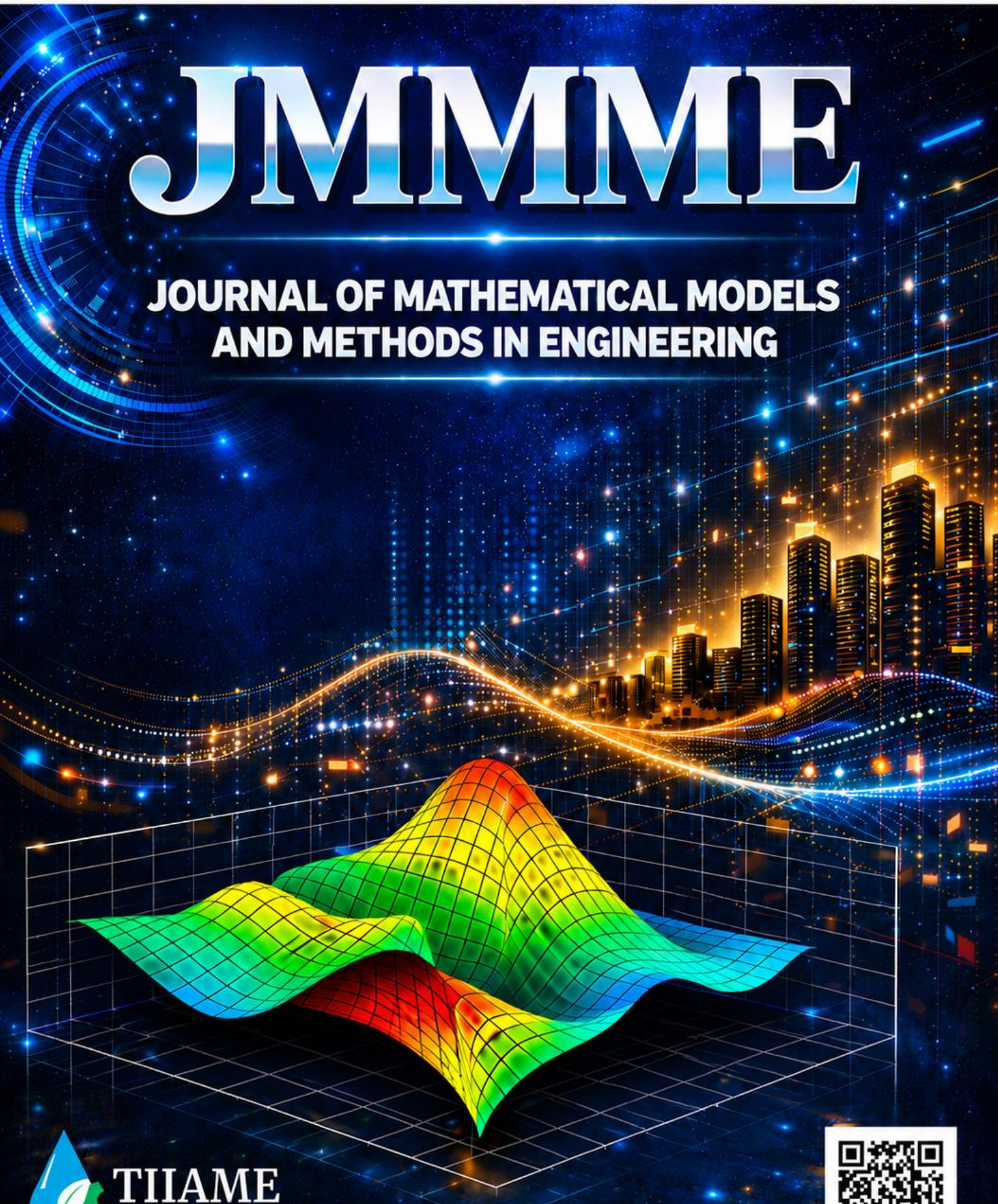
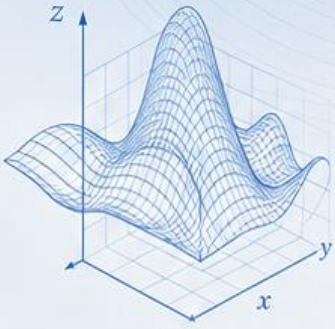
INTERNATIONAL STANDARD SERIAL NUMBER INTERNATIONAL CENTRE
Google Scholar
zenodo
doi
OpenAIRE
OPEN ACCESS

JMMME

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING



TIIAME
National Research University



JMMME

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING

$$\frac{dx}{dt} = Ax + b(t)$$

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u + f$$

$$\sum_{i=1}^{n} a_i x_i + b$$

$$f(x) = \int_a^b f(x)\,dx$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

JOURNAL INFORMATION

"Journal of Mathematical Models and Methods in Engineering" is an international, peer-reviewed, and open-access scientific journal established in the Republic of Uzbekistan by the National Research University "Tashkent Institute of Irrigation and Agricultural Mechanization Engineers" under Certificate No. 1648073 dated June 15, 2026.

$$\sum_{i=1}^{n} w_i x_i$$

The journal specializes in publishing scientific research on mathematical modeling, numerical methods, computational algorithms, optimization, and analytical approaches across various fields of engineering — including civil engineering, mechanical engineering, energy, hydraulic engineering, materials science, robotics, and environmental protection. It serves as a scientific platform for scholars, engineers, educators, and doctoral researchers to exchange theoretical and applied knowledge.



All submitted manuscripts undergo a double-blind peer review process and are published in strict compliance with high academic quality standards and publication ethics.



The journal has been published since June 15, 2026, once per quarter, that is, 4 times per year. Manuscripts are accepted in English and must be prepared in accordance with international IMRAD standards.

TIIAME
National Research University

FOUNDER	TIIAME National Research University
ADDRESS	39 Qori-Niyoziy Street, Mirzo Ulugbek District, Tashkent, Republic of Uzbekistan
PHONE	+998 (93) 397-12-39
WEBSITE	jmmme.uz
CERTIFICATE	No. 1648073, June 15, 2026
PUBLISHED	2026-06-15
LANGUAGE	English
FREQUENCY	Quarterly / 4 times per year
REVIEW	Double-blind peer review
STANDARD	International IMRAD standards

0 1 0 1
1 0 1 0
0 1 0 1
1 0 1 1
1 1 0 1
1 1 0 1
0 1 1 0
0 0 1 1
1 1 0 1
1 1 1 1
2 0 2 2
1 2 1 1

INDEXED AND DISCOVERABLE THROUGH

ISSN INTERNATIONAL STANDARD SERIAL NUMBER | Google Scholar | zenodo | doi | OpenAIRE | OPEN ACCESS

JMMME

TIIAME National Research University

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING

EDITOR-IN-CHIEF

Xudayarov Baxtiyar Alimovich

Doctor of Technical Sciences, Professor

TIIAME National Research University, Tashkent, Republic of Uzbekistan

ASSOCIATE EDITOR

Abdullayev Akmaljon Abdujalilovich

PhD, Associate Professor, Department of Higher Mathematics

TIIAME National Research University, Tashkent, Republic of Uzbekistan

COMPUTER OPERATOR AND DESIGNER

Sattarov Ulug'bek Umed o'g'li

EDITORIAL BOARD MEMBERS

Turayev Fozil Jurakulovich
Head of the Department of Higher Mathematics, Associate Professor
TIIAME National Research University, Tashkent, Republic of Uzbekistan

Ergashev Tukhtasin Gulomjonovich
Doctor of Physical and Mathematical Sciences, Professor
TIIAME National Research University, Tashkent, Republic of Uzbekistan

Eshkuvatov Zaynidin Karimovich
Doctor of Physical and Mathematical Sciences, Professor
TIIAME National Research University, Tashkent, Republic of Uzbekistan

Eshmatov Baxtiyor Xasanovich
Doctor of Physical and Mathematical Sciences, Associate Professor
TIIAME National Research University, Tashkent, Republic of Uzbekistan

INTERNATIONAL SUBJECT EDITORS

Andriy Verlan
Associate Professor, Department of Manufacturing and Civil Engineering, Norwegian University of Science and Technology (NTNU), Trondheim, Norway.

Dr. Mohd Kamal Mohd Nawawi
PhD, Associate Professor, School of Quantitative Sciences, Universiti Utara Malaysia, Kedah, Malaysia.

Vatin Nikolay Ivanovich
Doctor of Technical Sciences, Professor, Peter the Great St. Petersburg Polytechnic University.

Norihan Md Arifin
Professor of Applied Mathematics, Head of the Department of Mathematics and Statistics, Universiti Putra Malaysia, Selangor, Malaysia.

Sergey Leonov
Doctor of Engineering, Professor at National Technical University "Kharkiv Polytechnic Institute".

Amarjit Singh
PhD, Department of Mechanical Engineering, National Institute of Technology Hamirpur HP, India.

Vishesh Ranjan Kar
Assistant Professor, Department of Mechanical Engineering, National Institute of Technology Jamshedpur, India.

Muhammad Suzuri Hitam
Professor, Faculty of Computer Science and Mathematics, University Malaysia Terengganu, Malaysia.

Tkachenko Oleg Pavlovich
Doctor of Physical and Mathematical Sciences, Associate Professor, Leading Researcher at the Computing Center of the Far Eastern Branch of the Russian Academy of Sciences, Khabarovsk, Russia.

Gobithaasan Rudrusamy
Assoc. Prof. Dr. of School of Mathematical Sciences, University Sains Malaysia, Penang, Malaysia.

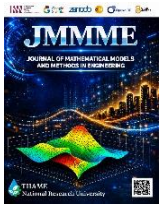
Prof. Dr. Muhammad Zaini Bin Ahmad
Doctor of Industrial Computing, Professor, Department of Mathematical Sciences, Faculty of Intelligent Computing, Universiti Malaysia Perlis, Malaysia.

Mahmoud Mohamed Selim Saleh
Doctor of Applied Mathematics, Professor, College of Science and Humanities, Prince Sattam bin Abdulaziz University, Al-Kharj, Saudi Arabia.

Pradeep G. Siddheshwar
M.Sc., Ph.D. Senior Professor of Mathematics, Bangalore 560 029, India.

Journal of Mathematical Models and Methods in Engineering

journal homepage:
https://jmmme.uz/index.php/journal



An Automatic Quadrature Scheme for Bounded and Unbounded Weighted Hypersingular Integrals on the Variable Interval

Z.K. Eshkuvatov[1*], H.X. Mamatova[2], R.D. Aloev[3], Sh. Ismail[4], O.S. Asadova[5]

[1,5]*Department of Higher Mathematics, Faculty of Land Resources and Cadaster, Tashkent Institute of Irrigation and Agricultural Mechanization Engineers, National Research University ("TIIAME", NRU), Tashkent, Uzbekistan.*
[2] *Department of Software Engineering and Cybersecurity, Fergana State Technical University (FSTU), Fergana, Uzbekistan.*
[3] *Faculty of Applied Mathematics and Intellectual Technologies, National University of Uzbekistan, Tashkent, Uzbekistan.*
[4] *Faculty of Science, Universiti Sains Islam Malaysia (USIM), Nigeri Sembilan, Malaysia.*

ARTICLE INFO

Article history:
Received 21 October 2026
Received in revised form 25 October 2026
Accepted 29 October 2026
Available online 1 November 2026

Keywords:

Automatic quadrature formula, Chebyshev polynomilas, Smoothness, Accuracy.

ABSTRACT

This note considers bounded and unbounded Hadamard-type hypersingular integrals of second-order singularity on an interval containing the singular point in its interior. The weight functions either vanish or become singular at the endpoints of the interval, and the density function is assumed to be sufficiently smooth. For each of the two types of integrals, automatic quadrature schemes are constructed. The approach is based on approximating the density function by truncated series of Chebyshev polynomials of the first and second kind, respectively, followed by the application of known analytical techniques for evaluating integrals of these polynomials. For the case where the limits of integration coincide with the endpoints of the standard interval, standard Gauss–Chebyshev quadrature formulas are also constructed.
Numerical experiments showed that the proposed methods are highly accurate in cases where the density is a polynomial or a rational function.

1. Introduction

The theory of singular integral equations (SIEs) with the integral in the sense of its principal value was originated almost simultaneously with the theory of Fredholm integral equations. In the early 20th century Hilbert (23 January 1862 – 14 February 1943) and Poincaré (9 April 1854 – 17 July 1912) examined SIEs while investigating different type of problems, viz, when Hilbert studying some boundary value problems of analytic function and Poincare was improving the theory of tides. Both theories are associated with numerous applications of singular and hypersingular integral equations, as well as with Riemann's boundary value problem. The Riemann's boundary value problem, singular, and hypersingular integral equations are widely used as basic techniques of mathematical modeling in physics: Quantum field theory (Bogolubov et al. [1]), theory of close and long-distance interaction (Brown and Jackson [2]), theory of elasticity and thermos-elasticity (Ioakimidis [3]), aerodynamics

[*1] *Corresponding author: Department of Higher Mathematics, Faculty of Land Resources and Cadaster, Tashkent Institute of Irrigation and Agricultural Mechanization Engineers, National Research University ("TIIAME", NRU), Tashkent, Uzbekistan.* E-mail: ezainidin@gmail.com

(Lifanov [4]), and electrodynamics (Lifanov et al. [5]), three-dimensional elasticity problems (Boykov [6]) and so on.

Let us consider integrals with strong singularity in general form

$$I_p(x,y;c,f)=\int_y^z \frac{f(t)}{w_r(t)(t-c)^p}dt,\ \ c\in(a,b),\ p\in N,, \tag{1}$$

where $w(t)$ is a weight function, $f(t)$ is the given differentiable function in certain order and x,y are variables with $a\le x<y\le b$.

Particularly, when $p=1$, Eq. (1) is called Cauchy type singular integrals (CSIs), if $p=2$ it is named Hadamard type SIs (or hypersingular integrals "HSIs") and when $p>2$ it is called super-singular integrals (SuperSIs). Main issue of this note is to find highly approximate solution of the SIs (1) for the case of $p=2$. To understand the problem well let us recall some definition of Cauchy-Hadamard finite-part integral. The concept of one-dimensional CPV integral is well introduced and widely used in applications. There are several definitions, mathematically equivalent for the finite-part integral in the literatures. One of them is the followings.

Definition 1 (Boykov et al. [6] and Zhang et al. [7]). The Hadamard principal sense of the integral (1) with $\mathrm{w(t)}=1$ is defined as the limit of the following expression

$$\int_a^b \frac{f(t)}{(t-s)^{p+1}}dt=lim_{\varepsilon\to\infty}\left\{\int_a^{s-\varepsilon}\frac{f(t)}{(t-s)^{p+1}}dt+\int_{a+\varepsilon}^b\frac{f(t)}{(t-s)^{p+1}}dt-\frac{2f^{(p-1)}(s)}{\varepsilon}\right\}, \tag{2}$$

where $p>1$ and $f(t)$ is said to be finite-part integrable (or integrable in the Hadamard sense) if the limit on the right-hand side of (2) exists.

It is known that differentiation of the CPV integrals

$$C(f,x)=\int_{-1}^1\frac{w(t)f(t)}{t-x}dt,\ \ x\in(-1,1), \tag{3}$$

yields

$$H(f,x)=\int_{-1}^1\frac{w(t)f(t)}{(t-x)^2}dt=\frac{d}{dx}\int_{-1}^1\frac{w(t)f(t)}{t-x}dt=\frac{d}{dx}C(f,x), \tag{4}$$

here, $H(f,x)$ is called hypersingular integrals (HSIs) and represents a natural extension of singular integrals in the CPVs.

Many problems in mechanics require the evaluation of integrals with a second order singularity. Particularly, Hui and Shia [8] developed a Gaussian quadrature formula using orthonormal polynomials for hypersingular integrals (4) and numerical experiments are carried out and compared with the quadrature formula developed by Kutt [9]. Chen [10] studied the plane elastic problem for a curved crack problem by means of the hypersingular integral equation approach. Nik Long and Eshkuvatov [11] proposed the complex variable function method and used to formulate the multiple curved crack problems into hypersingular integral equations. However Chan et. al. [12], addressed how the Hadamard finite part integral is used to solve the plane crack problems. Kim [13] considered the numerical evaluation of one sided finite part integrals of (4) with $w(t)=(1-t)^\alpha(1+t)^\beta,\beta>-1,-1<\alpha<0$. Chan et. al. [14] investigated hypersingular integrals of the form

$$I_\alpha(T_n,m,r)=\oint_{-1}^1\frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^\alpha}ds,\qquad |r|<1,m\in N$$

$$I_\alpha(U_n,m,r)=\oint_{-1}^1\frac{U_n(s)(1-s^2)^{m-1/2}}{(s-r)^\alpha}ds,\ |r|<1,m\in N \tag{5}$$

where $T_n(x)$ and $U_n(x)$ are Chebyshev polynomials of the first and second kind, respectively, and the exact solution are derived for the cases $\alpha=\{1,2,3,4\}$ and $m=\{0,1,2,3,4\}$.

Obayis et al. [15] presented an automatic quadrature scheme (AQS) based on Clenshaw–Curtis–Chebyshev quadrature methods for the evaluation of HSIs of the form

$$Q_i(f,x) = \int_{-1}^{1} \frac{w(t)f(t)}{(t-x)^2}\,dt\,,\ \ x \in (-1,1)\,,\ i = \{0,1,2\} \tag{6}$$

where $w_0(t) = 1,\ w_1(t) = \sqrt{1-t^2}$ and $w_2(t) = \frac{1}{\sqrt{1-t^2}}$ are the weight functions, and the given function $f(t)$ imperative to have certain smoothness properties. Particular attention is paid to error estimate of the developed AQS obtained in the class of functions $C^{N+2,\alpha}$ for $i = 0$ case, which converges to the exact solution very fast by increasing the knot points. Bichi et al. [16] considered the HSIs (6) with the weights $w_3(t) = \sqrt{\frac{1-t}{1+t}}$ and $w_4(t) = \sqrt{\frac{1+t}{1-t}}$. For the construction of AQS, the density function $f(t)$ is approximated by third and fourth kind Chebyshev polynomials. Several examples are provided to validate the proposed methods. Eshkuvatov et al. [17] constructed hybrid quadrature formula for singular integral of Cauchy type with four types. Shadimetov and Akhmedov [18] developed optimal quadrature formula for approximate solution of the 1st kind Fredholm singular integral equations and Akhmedov and Pirnapasov [19] constructed optimal approximation techniques for solving Hadamard-type integral equations. Eshkuvatov et al. [20-21] have used different type of methods to solve singular integral equations of the first kind.

In this note, we have developed the AQS based on Chebyshev-Gauss quadrature formula for the hypersingular integrals of the form

$$H_r(h,c) = \frac{w_r(c)}{\pi}\int_y^z \frac{f(t)}{w_r(t)(t-c)^2}\,dt\,,\ \ c \in (-1,1),\ -1 \le y < z \le 1, \mathrm{r} = \{1,2\}, \tag{7}$$

where $h(t)$ is a smooth function and

$$w_1(t) = \sqrt{1-t^2},\quad w_2(t) = \frac{1}{\sqrt{1-t^2}}. \tag{8}$$

Particularly, in Section 2, the mathematical concepts of the first and second kind Chebyshev polynomials and Cauchy-Hadamard SIs are presented. In Section 3, the detail construction of AQS for the bounded $r = 1$ and unbounded $r = 2$ HSIs are given. Numerical examples are provided in Section 4, followed by the conclusion and acknowledgment in Section 5.

2. Mathematical concepts of the first and second kind chebyshev polynomials and singular integrals

In this section we recall some concepts of the Chebyshev polynomials and singular integrals.

Definition 2 (Mason & Handscomb [22]): The Chebyshev polynomials of the first kind $\mathrm{T_m(t)}$ and of the second kind $\mathrm{U_m(t)}$ are the polynomials of degree m on the interval $[-1,1]$ defined by

$$\begin{cases} T_m(t) = cos(m\theta), \\ U_m(t) = \frac{sin(m+1)\theta}{sin\,\theta}, \end{cases} \tag{9}$$

where $t = cos\theta, 0 \le \theta \le \pi$.
Both polynomials

$$P_{r,m}(t) = \begin{cases} T_m(t), & r = 1, \\ U_m(t), & r = 2, \end{cases} \tag{10}$$

satisfy the recurrence relations

$$\begin{cases} P_{r,m}(t) = 2tP_{r,m-1}(t) - P_{r,m-2}(t),\ \ m = 2,3,\cdots, r = \{1,2\} \\ P_{0,0}(t) = 1,\ \ P_{1,1}(t) = t, \qquad P_{2,1}(t) = 1,\ \ P_{2,1} = 2t. \end{cases} \tag{11}$$

The function $P_{r,m}$ satisfy the orthogonality properties

$$\mu_{m,n}^{r}=\frac{1}{\pi}\langle P_{m,r},P_{n,r}\rangle_r=\begin{cases}0,\ m\neq n,\ r=\{1,2\},\\ 1,\ m=n=0,\ r=1,\\ \frac{1}{2},\ m=n\neq 0,\ r=1,\\ \frac{1}{2},\ m=n,\ r=2.\end{cases} \tag{12}$$

with respect to inner product

$$\langle f,g\rangle=\int_{-1}^{1}w_r(t)f(t)g(t)dt, \tag{13}$$

where weights $w_r(t), r=\{1,2\}$ are defined by (8).
In Mason and Handscomb [22] shown the relationship of Chebyshev polynomials

$$\begin{aligned}T_{k+1}(t)&=\frac{1}{2}[U_{k+1}(t)-U_{k-1}(t)],\\ (1-t^2)U_k(t)&=\tfrac{1}{2}[T_k(t)-T_{k+2}(t)],\end{aligned} \tag{14}$$

and differentiation of Chebyshev polynomials of the first and second kind are

$$\begin{aligned}\frac{d}{dx}T_k(t)&=kU_{k-1}(t),\\ (1-t^2)\frac{d}{dt}U_k(t)&=\tfrac{1}{2}[(k+1)U_{k-2}(t)-(k-1)U_k(t)].\end{aligned} \tag{15}$$

The following are Hilbert type integral transform
Lemma 1 (Mason & Handscomb [22]).

$$\int_0^{\pi}\frac{\cos m\theta}{\cos\theta-\cos\alpha}d\theta=\pi\frac{\sin m\alpha}{\sin\alpha}, \tag{16}$$

for any $\alpha\in[0,\pi]$, $m=1,2,3,\cdots$.
Lemma 2 (Mason and Handscomb [22]).

$$\int_0^{\pi}\frac{\sin m\theta\sin\theta}{\cos\theta-\cos\alpha}d\theta=-\pi\cos m\alpha, \tag{17}$$

for any $\alpha\in[0,\pi]$, $m=1,2,3,\cdots$.
The bound for some important integrals are established
Theorem 1 (Mason and Handscomb [22]). *As a CPV integral, we have*

$$S_{j,r}(x)=\frac{1}{\pi}\int_{-1}^{1}\frac{w_r(t)P_{j,r}(t)}{t-x}dt=\begin{cases}U_{j-1}(x),\ r=1,\\ -T_{j+1}(x)\ \ r=2,\end{cases} \tag{18}$$

It is shown in Israilov [18], that

$$\int\frac{dt}{\sqrt{1-t^2}(t-c)}=\frac{1}{\sqrt{1-c^2}}\ln\left|\frac{t\sqrt{1-c^2}-x\sqrt{1-t^2}}{\sqrt{1-c^2}+\sqrt{1-t^2}}\right|+C, \tag{22}$$

where C is a constant.

3. Automatic quadrature schemes for bounded and unbounded hypersingular integrals

In this section, we construct two types of AQS. In Section 3.1, we construct AQS for HSIs on the variable interval and in Section 3.2, AQS is constructed on the fixed interval.

3.1 Construction of AQS for HSIs on the variable interval

To construct AQS for HSIs (5) we do the following transformation

$$H_r(h,y,z,c)=\frac{w_r(c)}{\pi}\frac{d}{dc}\left[\int_y^z\frac{h(t)-h(c)}{w_r(t)(t-c)}dt+h(c)\int_y^z\frac{dt}{w_r(t)(t-c)}\right]$$
$$=\frac{w_r(c)}{\pi}\frac{d}{dc}\left[I_{r,1}(c,y,z)+I_{r,2}(c,y,z)\right],\ r=\{1,2\},\tag{23}$$

where

$$I_{r,1}(c,y,z)=\int_y^z\frac{h(t)-h(c)}{w_r(t)(t-c)}dt\,,r=\{1,2\},$$
$$I_{r,2}(c,y,z)=h(c)\int_y^z\frac{dt}{w_r(t)(t-c)}dt\,,\quad r=\{1,2\}.\tag{24}$$

The case of $r=1$. From (7)-(8) and (23), we have

$$H_1(h,y,z,c)=\frac{\sqrt{1-c^2}}{\pi}\frac{d}{dc}\left[I_{1,1}(y,z,c)+I_{1,2}y,z,c\right],\tag{25}$$

where

$$I_{1,1}(y,z,c)=\int_y^z\frac{h(t)-h(c)}{\sqrt{1-t^2}(t-c)}dt\,,I_{1,2}(y,z,c)=h(c)\int_y^z\frac{dt}{\sqrt{1-t^2}(t-c)}.\tag{26}$$

To evaluate SIs in (25)-(26), let the function $h(t)$ is approximated by the Chebyshev polynomials

$$h(t)\approx P_{r,N}(t)=\begin{cases}\sum_{m=0}^{N}{}''a_{1m}T_m(t),r=1,\\ \sum_{m=0}^{N}{}''a_{2m}U_m(t),r=2.\end{cases}\tag{27}$$

where " denotes the first and last term are halved. The techniques of Hasegava & Torri [24] will be used to evaluate the first integral in Eq. (26) as follows

$$\frac{h(t)-h(c)}{(t-c)}\approx\frac{P_{1N}(t)-P_{1N}(c)}{(t-c)}=\sum_{k=0}^{N-1}{}'b_{1k}T_k(t).\tag{28}$$

To find the coefficients b_{1k} we rewrite Eq. (28) in the form

$$P_{1N}(t)-P_{1N}(c)=\sum_{k=0}^{N-1}{}'b_{1k}T_k(t)\,(t-c)\tag{29}$$

It is known that

$$tT_k(t)=\frac{1}{2}[T_{k+1}(t)+T_{k-1}(t)],k=1,\dots,N-1.\tag{30}$$

Due to (27)-(29) and restructuring right side of (29) using (30), we arrive at

$$\sum_{k=1}^{N}a_{1k}(T_k(t)-T_k(c))'=\frac{1}{2}\sum_{k=1}^{N}[b_{1k-1}-2cb_{1k}+b_{1k+1}]T_k(t),\tag{31}$$

with $b_{1N}=b_{1N+1}=0$ and relationship (29) is valid for any $t\in[-1,1]$, particularly for $t=c$ gives

$$0=\frac{1}{2}\sum_{k=1}^{N}[b_{1k-1}-2cb_{1k}+b_{1k+1}]T_k(c)+\frac{1}{2}[b_{11}-b_{10}c]T_0(c).\tag{32}$$

Subtracting the zero term given by (32) on the right side of (31), we obtain

$$\sum_{k=1}^{N}a_{1k}(T_k(t)-T_k(c))=\frac{1}{2}\sum_{k=1}^{N}[b_{1k-1}-2cb_{1k}+b_{1k+1}]\big(T_k(t)-T_k(c)\big).\tag{33}$$

By equating similar terms both sides of Eq. (32), we arrive at three diagonal system of algebraic equation

$$\begin{cases}b_{1k+1}-2cb_{1k}+b_{1k-1}=2a_{1k},k=1,\cdots,N-1\\ b_{1N-1}=a_{1N},b_{1N}=0.\end{cases}\tag{34}$$

Double sweep method is used to find the coefficients $b_k,k=0,\dots,N-1$. Once we solve Eq. (34) for b_k it will be function of c therefore for the future we use the notations $b_{1k}=b_{1k}(c)$. It is known that

$$\int \frac{T_k(t)}{\sqrt{1-t^2}}dt = -\frac{1}{2k\sqrt{1-t^2}}\left[T_{k-1}(t) - T_{k+1}(t)\right] + c, \tag{35}$$

To evaluate the first integral of (26) for $r = 1$ with $k = 1,2,\cdots,N$ the (35) is used to get

$$\begin{aligned} I_{1,1}(c,y,z) &\cong \sum_{k=0}^{N-1} b_{1k}(c)\int_y^z \frac{T_k(t)}{\sqrt{1-t^2}}dt \\ &= \frac{b_{10}(c)}{2} arcsin(t)_y^z - \sum_{k=1}^{N-1}\frac{b_{1k}(c)}{2k}\frac{1}{\sqrt{1-t^2}}(T_{k-1}(t) - T_{k+1}(t))_y^z \\ &= \frac{b_{10}(c)}{2} arcsin(t)|_y^z - \frac{1}{\sqrt{1-t^2}}\left(\frac{b_{11}(c)}{2}T_0(t) + \frac{b_{12}(c)}{4}T_1(t)\right)_y^z \\ &\quad - \frac{1}{2}\left[\sum_{k=2}^{N-2}\left(\frac{b_{1k+1}(c)}{k+1} - \frac{b_{1k-1}(c)}{k-1}\right)\frac{T_k(t)}{\sqrt{1-t^2}}\right]_y^z + \left[\frac{b_{1N-2}(c)}{2(N-2)}\frac{T_{N-1}(t)}{\sqrt{1-t^2}}\right. \\ &\quad \left. + \frac{b_{1N-1}(c)}{2(N-1)}\frac{T_N(t)}{\sqrt{1-t^2}}\right]_y^z = I_{1N,1}(c,y,z). \end{aligned} \tag{36}$$

For the second term of (26), we use (22) to get

$$I_{1,2}(c,y,z) = \frac{h(c)}{\sqrt{1-c^2}}\left[ln\frac{t\sqrt{1-c^2} - c\sqrt{1-t^2}}{\sqrt{1-c^2}+\sqrt{1-t^2}}\right]_y^z. \tag{37}$$

From (36)-(37) and (25) it follows that

$$\begin{aligned} H_1(f,y,z,c) \cong H_{1N}(f,y,z,c) &= \frac{\sqrt{1-c^2}}{\pi}\frac{d}{dc}\left[I_{1N,1}(c,y,z) + I_{1,2}(c,y,z)\right] \\ &= \frac{\sqrt{1-c^2}}{\pi}\frac{d}{dc}\left\{\frac{b_{10}(c)}{2} arcsin(t)|_y^z - \frac{1}{\sqrt{1-t^2}}\left[\frac{b_{11}(c)}{2}T_0(t) + \frac{b_{12}(c)}{4}T_1(t)\right]\right. \\ &\quad - \frac{1}{2\sqrt{1-t^2}}\left[\sum_{k=2}^{N-2}\left(\frac{b_{1k+1}(c)}{k+1} - \frac{b_{1k-1}(c)}{k-1}\right)T_k(t)\right]_y^z \\ &\quad + \frac{1}{2\sqrt{1-t^2}}\left[\frac{b_{1N-2}(c)}{(N-2)}T_{N-1}(t) + \frac{b_{1N-1}(c)}{(N-1)}T_N(t)\right]_y^z \\ &\quad \left. + \left[\frac{h(c)}{\sqrt{1-c^2}} ln\frac{t\sqrt{1-c^2} - c\sqrt{1-t^2}}{\sqrt{1-c^2}+\sqrt{1-t^2}}\right]_y^z\right\}. \end{aligned} \tag{38}$$

where $b_{1k}(c), k = 0,1,\ldots,N-1$ are defined by (34).

For the case of $r = 2$. From (7) and (23) it follows that

$$H_2(f,y,z,c) = \frac{1}{\pi\sqrt{1-c^2}}\frac{d}{dc}\left[I_{2,1}(y,z,c) + I_{2,2}(y,z,c)\right], \tag{39}$$

where

$$I_{2,1}(y,z,c) = \int_y^z \frac{\sqrt{1-t^2}[h(t)-h(c)]}{t-c}dt, \quad I_{2,2}(y,z,c) = h(c)\int_y^z \frac{\sqrt{1-t^2}}{t-c}dt. \tag{40}$$

For the second integral $I_{2,2}$ in Eq. (40), we use (22) and perform simple operations to obtain

$$I_{2,2}(y,z,c) = h(c)\int_y^z \frac{(1-t^2)}{\sqrt{1-t^2}(t-c)}dt$$

$$= h(c)\sqrt{1-c^2}\left[ln\frac{t\sqrt{1-c^2}-c\sqrt{1-t^2}}{\sqrt{1-c^2}+\sqrt{1-t^2}}\right]_y^z + h(c)\left[\sqrt{1-t^2} - c\cdot arcsin(t)\right]_y^z. \quad (41)$$

For the evaluation of $I_{2,1}$ in Eq. (40), let

$$h(t) \cong P_{2N}(t) = \sum_{m=0}^{N}{}'a_{2m}U_m(t), \quad (42)$$

then

$$I_{2,1}(y,z,c) = \int_y^z \frac{\sqrt{1-t^2}(h(t)-h(c))}{t-c}dt \cong \int_y^z \frac{\sqrt{1-t^2}(P_{2N}(t)-P_{2N}(c))}{t-c}dt \quad (43)$$

To evaluate the integrand function in Eq. (43), we consider

$$\frac{P_{2N}(t)-P_{2N}(c)}{t-c} = \sum_{k=0}^{N-1} b_{2k}U_k(t), \quad (44)$$

and using

$$tU_k(t) = \frac{1}{2}[U_{k+1}(t) + U_{k-1}(t)], k = 1,\cdots N-1,$$

with the condition $b_{2N} = b_{2N+1} = 0$, as well as doing the same approach as we did for getting recurrence relation for b_{1k} we arrive at

$$\begin{cases} b_{2k+1} - 2cb_{2k} + b_{2k-1} = 2a_{2k}, k = 1,\cdots,N-1 \\ b_{2N-1} = a_{2N}, b_{2N} = 0. \end{cases} \quad (45)$$

To evaluate $I_{2,1}$, the following exact integration formula

$$\int\sqrt{1-t^2}U_k(t)dt = \frac{\sqrt{1-t^2}}{2}\left[\frac{U_{k+1}(t)}{k+2} - \frac{U_{k-1}(t)}{k}\right],\ k = \{1,2,\mathrm{K}\} \quad (46)$$

and Eqs. (22) is used to yield

$$I_{2,1}(c,y,z) \cong I_{2N,1}(c,y,z) = \frac{b_{20}(c)}{2}\int_y^z \sqrt{1-t^2}U_0(t)dt + \sum_{k=1}^{N-1} b_{2k}(c)\int_y^z \sqrt{1-t^2}U_k(t)dt$$

$$= \frac{b_{20}(c)}{4}\left[t\sqrt{1-t^2} + arcsin(t)\right]_y^z + \sum_{k=1}^{N-1}\frac{b_{2k}(c)}{2}\sqrt{1-t^2}\left[\frac{U_{k+1}(t)}{k+2} - \frac{U_{k-1}(t)}{k}\right]_y^z$$

$$= \frac{b_{20}(c)}{4}\left[t\sqrt{1-t^2} + arcsin(t)\right]_y^z - \frac{\sqrt{1-t^2}}{2}\left[b_{21}(c)U_0(t) + \frac{b_{22}(c)}{2}U_1(t)\right]_y^z$$

$$+ \frac{\sqrt{1-t^2}}{2}\left[\frac{b_{2N-2}(c)}{N}U_{N-1}(t) + \frac{b_{2N-1}(c)}{N+1}U_N(t)\right]_y^z$$

$$+ \left[\sum_{k=2}^{N-2}\left(\frac{b_{2k-1}(c)}{k+1} - \frac{b_{2k+1}(c)}{k+1}\right)U_k(t)\right]_y^z. \quad (47)$$

Substituting (43) and (48) into (41) yields

$$H_{2N}(f,y,z,c) = \frac{1}{\pi\sqrt{1-c^2}}\frac{d}{dc}\{I_{1,2N}(y,z,c)\} + I_{2,2}(y,z,c)$$

$$= \frac{1}{\pi\sqrt{1-c^2}}\frac{d}{dc}\left\{\frac{b_{20}(c)}{4}\left[t\sqrt{1-t^2} + arcsin(t)\right]_y^z\right.$$

$$\left. - \frac{\sqrt{1-t^2}}{2}\left[b_{21}(c)U_0(t) + \frac{b_{22}(c)}{2}U_1(t)\right]_y^z\right.$$

$$+\frac{\sqrt{1-t^2}}{2}\left[\frac{b_{2N-2}(c)}{N}U_{N-1}(t)+\frac{b_{2N-1}(c)}{N+1}U_N(t)\right]_y^z$$
$$+\left[\sum_{k=2}^{N-2}\left(\frac{b_{2k-1}(c)}{k+1}-\frac{b_{2k+1}(c)}{k+1}\right)U_k(t)\right]_y^z$$
$$+h(c)\left[\sqrt{1-t^2}-c\,arcsin(t)\right]_y^z+h(c)\sqrt{1-c^2}\left\{\left[ln\frac{t\sqrt{1-c^2}-c\sqrt{1-t^2}}{\sqrt{1-c^2}+\sqrt{1-t^2}}\right]_y^z. \quad (48)$$

Now, we will deriv Gauss-Chebyshev quadtature formula on the fixed interval.

3.2 AQS for HSIs on the fixed interval

To construct AQS for the bounded and unbounded HSIs (5) when $y=-1, z=1$, the density function $h(t)$ is approximated with series of truncated Chebyshev polynomials. Substitute (6) into Eq. (5) and use Mason's theorem [17](10) to obtain

$$H_r(h,x)\approx H\left(P_{r,N},x\right)=\frac{w_r(t)}{\pi}\int_{-1}^{1}\frac{P_{r,N}(t)}{w_i(t)(t-x)^2}dt$$
$$=\begin{cases}\sqrt{1-x^2}\sum_{m=0}^{N}{}''a_{1m}\frac{d}{dx}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{T_m(t)}{\sqrt{1-t^2}(t-x)}dt\right], r=1\\ \frac{1}{\sqrt{1-x^2}}\frac{d}{dx}\sum_{m=0}^{N}{}''a_{2m}\frac{d}{dx}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}U_m(t)}{(t-x)}dt\right], r=2\end{cases}$$
$$=\begin{cases}\sqrt{1-x^2}\frac{d}{dx}\sum_{m=0}^{N}{}''a_{1m}U_{m-1}(x), r=1,\\ \sqrt{1-x^2}\frac{d}{dx}\sum_{m=0}^{N}{}''a_{2m}T_{m+1}(x), r=2,\end{cases} \quad (49)$$

By applying the orthogonality relations (12), the continuous coefficients $a_{r,m}, i=\{1,2\}$ are

$$a_{r,m}=\begin{cases}\frac{2}{\pi}\int_{-1}^{1}\frac{h(t)T_m(t)dt}{\sqrt{1-t^2}}, & r=1,\\ \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}h(t)U_m(t)dt, & r=2.\end{cases} \quad (50)$$

For the discrete coefficients $a_{r,m}$, we choose the collocation points at the zeros of T_{N+1} and U_{N+1} namely

$$t=t_{1k}=cos\frac{(k+1/2)\pi}{N+1}=cos\frac{(2k+1)\pi}{2N+2}, k=0,1,2,\cdots,N,$$
$$t=t_{2k}=cos\frac{(k\pi)}{N+1}, k=1,2,\cdots,N. \quad (51)$$

Then, using orthogonality conditions (12) and relation (51) for $r=\{1,2\}$, and applying composite trapezoidal rule, we arrive at

$$a_{r,m}=\begin{cases}\frac{2}{N+1}\sum_{k=0}^{N}T_m(t_k)h(t_k), & r=1,\\ \frac{2}{N+1}\sum_{k=0}^{N}(1-t_k^2)U_m(t_k)h(t_k), & r=2.\end{cases} \quad (52)$$

4. Numerical results and discussion

We very often use the following Table to calculate the power of Chebyshev polynomials

Table 1: Chebyshev polynomials of the first and second kind

n	$T_n(x)$	$U_n(x)$
0	1	1
1	x	$2x$
2	$2x^2-1$	$4x^2-1$
3	$4x^3-3x$	$8x^3-4x$
4	$8x^4-8x^2+1$	$16x^4-12x^2+1$
5	$16x^5-20x^3+5x$	$32x^5-32x^3+6x$
6	$32x^6-48x^4+18x^2-1$	$64x^6-80x^4+24x^2-1$

In this section, we evaluate the hypersingular integrals (5) for different choices of $h(t)$.

Example 1: *Consider the HSI of the form*

$$H_1(h,c)=\frac{\sqrt{1-c^2}}{\pi}\int_{-0.9}^{0.9}\frac{h(t)}{\sqrt{1-t^2}(t-c)^2}dt \tag{53}$$

where $h(t)=t^2-1$.
Exact solution of (53) is

$$H_1(h,c)=\frac{\sqrt{1-c^2}}{\pi}\left[2.239539+\frac{c}{\sqrt{1-c^2}}ln\left|\frac{0.9\sqrt{1-c^2}-c\cdot 0.435889}{0.9\sqrt{1-c^2}+c\cdot 0.435889}\right|+\frac{0.7846018098}{0.81-c^2}\right] \tag{54}$$

To obtain the approximate solution we choose $N=5$ and from (6) we get

$$h(t)\cong P_{1,5}(t)=\sum_{m=0}^{5}\quad a_{1m}T_m(t), \tag{55}$$

and due to orthogonality conditions it follows that

$$a_{10}=-1, a_{11}=0, a_{12}=\frac{1}{2}, a_{1k}=0, k=\{3,4,5\}.$$

With these values of a_{1k}, we find from (55) that $P_{1,5}(t)=h(t)=t^2-1$.
To find the coefficients of b_k, we use the recurrence relation (11) and obtain that

$$b_{10}=2c, b_{11}=1, b_{1k}=0, k=\{2,3,4,5\}. \tag{56}$$

To determine the approximate solution for $N=5$, we use Eq. (38) which yields

$$\begin{aligned}H_{1,5}(f,y,z,c)\cong{}&\frac{\sqrt{1-c^2}}{\pi}\frac{d}{dc}\left\{b_{10}(c)\int_{-0.9}^{0.9}\frac{T_0(t)}{\sqrt{1-t^2}}dt+b_{11}(c)\int_{-0.9}^{0.9}\frac{T_1(t)}{\sqrt{1-t^2}}dt\right\}\\&-\frac{d}{dc}\sqrt{1-c^2}\left[ln\frac{t\sqrt{1-c^2}-c\sqrt{1-t^2}}{\sqrt{1-c^2}+\sqrt{1-t^2}}\right]_{-0.9}^{0.9}\\&=\frac{\sqrt{1-c^2}}{\pi}\left[2.239539+\frac{c}{\sqrt{1-c^2}}ln\left|\frac{0.9\sqrt{1-c^2}-c\cdot 0.435889}{0.9\sqrt{1-c^2}+c\cdot 0.435889}\right|\right.\\&\left.+\frac{0.7846018098}{0.81-c^2}\right]\end{aligned}$$

which is identical with exact solution (56). Thus, error term of approximate solution is exactly equal to zero for any singular point c inside $c \in [y,z] = [-0.9,0.9]$ and outside of the interval $c \in (-1,-0,9) \cup (0.9,1)$.

Example 2: Consider the following HSIs

$$H_1(h,c) = \frac{\sqrt{1-c^2}}{\pi} \int_{-0.9}^{0.9} \frac{h(t)}{\sqrt{1-t^2}(t-c)^2} dt \tag{57}$$

where $\mathrm{h(t)} = \frac{10t}{1+t^2}$. Its exact solution is

$$H_1(h,c) = \frac{\sqrt{1-c^2}}{\pi} \frac{1}{c^2+1} \left[-\frac{35.0964735c^6}{c^2+1} - \frac{7.8460180983(c^3+1)}{(c^2-0.81)(c^4-1)} + \frac{-20c^6+26.2c^4-18.1c^2+8.1}{\sqrt{1-c^2}(c^2-0.81)(c^4-1)} + ln \left| \frac{0.9\sqrt{1-c^2}-c\sqrt{1-0.9^2}}{0.9\sqrt{1-c^2}+c\sqrt{1-0.9^2}} \right| \right]. \tag{58}$$

From (27), it follows that

$$h(t) \cong P_{1,5}(t) = \sum_{m=0}^{5} a_{1m} T_m(t). \tag{59}$$

Due to orthogonality conditions (12), we have

$$a_{10} = 0, a_{11} = 20 - 10\sqrt{2}, a_{12} = 0, a_{13} = 70\sqrt{2} - 100,$$
$$a_{14} = 0, \quad a_{15} = -410\sqrt{2} + 580.$$

With these values a_{1k}, we find that

$$P_{1,5}(t) = (20 - 10\sqrt{2})T_1(t) + (70\sqrt{2} - 100)T_3(t) + (-410\sqrt{2} + 580)T_5(t).$$

To find the coefficients of b_{1k}, we use the recurrence relation (13)

$$b_{10} = -6560\sqrt{2}c^4 + 9280c^4 + 5480c^2\sqrt{2} - 7760c^2 - 570\sqrt{2} + 820,$$
$$b_{11} = -3280\sqrt{2}c^3 + 4640c^3 + 1920c\sqrt{2} - 2720c,$$
$$b_{12} = -1640c^2\sqrt{2} + 2320c^2 + 550\sqrt{2} - 780,$$
$$b_{13} = -20c\left(41\sqrt{2} - 58\right),$$
$$b_{14} = -410\sqrt{2} + 580,$$
$$b_{15} = 0.$$

With this values of b_{1k} we obtain

$$H_{1N}(h,y,z,c) \cong \frac{\sqrt{1-c^2}}{\pi} \frac{d}{dc} \left\{ \frac{b_{10}(c)}{2} \int_{-0.9}^{0.9} \frac{T_0(t)}{\sqrt{1-t^2}} dt + \sum_{k=1}^{5} b_{1k}(c) \int_{-0.9}^{0.9} \frac{T_k(t)}{\sqrt{1-t^2}} dt - \left[\frac{1}{\sqrt{1-c^2}} ln \frac{t\sqrt{1-c^2} - c\sqrt{1-t^2}}{\sqrt{1-c^2} + \sqrt{1-t^2}} \right]_{-0.9}^{0.9} \right\}$$
$$= -15.9507c^{17} + 80.917c^{15} - 187.175c^{13} + 269.998c^{11}$$
$$-285.2952c^9 + 247.8370c^7 - 189.48924c^5 + 126.663107c^3$$
$$- 63.34340835c + 1.24181c^{15} - 5.65799c^{13} + 11.68508c^{11}$$
$$- 15.06605c^9 + 14.36167c^7 - 11.317907c^5 + 7.6092977c^3$$
$$- 3.807433999c + 0.168421c^{13} - 0.679521c^{11} + 1.233708c^9$$
$$- 1.400012c^7 + 1.181594c^5 - 0.8077474c^3$$
$$+0.404980206c + \frac{0.78460180983732123940}{c^2-0.81} - \frac{c}{\sqrt{1-c^2}} ln \left| \frac{0.9\sqrt{1-c^2}-c\sqrt{1-0.9^2}}{0.9\sqrt{1-c^2}+c\sqrt{1-0.9^2}} \right| \tag{60}$$

Approximate solution for $N = \{5,10\}$ is illustrated in the following tables.

N	$c \in (y,z)$	Exact solution	AQS (38), for $r=1$	Absolute error
5	-0.8999	15884.554833152305273	15884.838478840740161	0.283645688434888
	-0.5961	7.7351364933288352075	7.7382576206619290187	0.031211273330938112
	-0.0121	0.209738218665095535 9	0.3578644715655138761	0.1481262529004183402
	0.0121	-0.209738218665095535 8	0.0327674189590749016 1	0.2425056376241704374
	0.5961	-7.7351364933288352075	-7.5302355131611660810	0.2049009801676691265
	0.8999	-15884.554833152305274	-15884.798913210793020	0.244080058487746
N	$\notin (y,z)$	Exact solution	AQS (38), for $r=2$	Absolute error
5	-0.901	-1543.0324180398803894	-1542.7502240225545903	0.282194017325799181
	-0.999	-1.0171534671885602186	-.99237056208652672090	0.024782905102033497 7
	0.901	1543.0324180398803894	1542.7892646964716174	0.2431533434087720
	0.999	1.0171534671885602471	0.99331025704250122617	0.023843210146059020 9

Table 2: Numerical results of HSIs (5) with $w_1(t)=\sqrt{1-t^2}$ for $N=5$.

N	$c \in (y,z)$	Exact solution	AQS (38), for $r=1$	Absolute error
10	-0.8999	15884.44438899821518	15884.55483315230596	0.11044415409077921
	-0.5961	7.451110979092140179	7.7351364933288352073	0.0284025514236695028
	-0.0121	0.196552383281141388	0.2097382186650955359	0.0131858353839541478
	0.0121	-.1965523832811413881	-.2097382186650955358	0.0131858353839541477
	0.5961	-7.451110979092140179	-7.735136493328835207	0.0284025514236695028
	0.8999	-15884.42423318385134	-15884.55483315230596	0.130599968454619
N	$c \notin (y,z)$	Exact solution	AQS (38), for $r=1$	Absolute error
10	-0.901	-1543.162221258282229	-1543.032418039880386	0.123803218401843233
	-0.999	-1.029342300107515805	-1.017153467188560232	0.0121888329189555730
	0.901	1543.162221258282229	1543.0324180398803859	0.1238032184018432
	0.999	1.029342300107515805	1.0171534671885602461	0.0121888329189555588

Table 3: Numerical results of HSIs (5) with $w_1(t)=\sqrt{1-t^2}$ for $N=10$.

Example 3: *Consider the HSIs of the form*

$$H_2(h,c)=\frac{1}{\pi\sqrt{1-c^2}}\int_{-0.9}^{0.9}\frac{\sqrt{1-t^2}h(t)}{(t-c)^2}dt \tag{61}$$

where $h(t)=\frac{10t}{1+t^2}$. Exact solution of Eq. (61) is

$$H_2(h,c)=\frac{1}{\pi\sqrt{1-c^2}}\left[-\frac{25.402166434635424044c}{(c^2+1)^2}\right.$$

$$+\frac{1}{(c^2+1)^2(c^2-0.81)\sqrt{1-c^2}}$$
$$+\sqrt{1-c^2}(-36944762501566c^3+44.12655038432461)$$
$$+\frac{-30c^4+34.3c^2-8.1}{\sqrt{1-c^2}}ln\left|\frac{0.9\sqrt{1-c^2}-c\sqrt{1-0.9^2}}{-0.9\sqrt{1-c^2}-c\sqrt{1-0.9^2}}\right|\Big] \quad (62)$$

To find the approximate solution we choose $N=5$ and use AQS (64). From (6), it follows that

$$h(t)\cong P_{2,5}(t)=\sum_{m=0}^{5} a_{2m}U_m(t).$$

Due to orthogonality conditions, we have

$$a_{20}=0, a_{21}=-40\sqrt{2}+60, a_{22}=0, a_{23}=240\sqrt{2}-340,$$
$$a_{24}=0, a_{25}=-1400\sqrt{2}+1980.$$

In order to find the coefficients of $b_{k,2}$, we use the recurrence relation (13) which yields

$$b_{20}=-22400\sqrt{2}c^4+31680c^4+18720c^2\sqrt{2}-2680c^2-1960\sqrt{2}+2780,$$
$$b_{21}=-11200\sqrt{2}c^3+15840c^3+6560c\sqrt{2}-9280c,$$
$$b_{22}=-5600c^2\sqrt{2}+7920c^2+1880\sqrt{2}-2660,$$
$$b_{23}=-40c(70\sqrt{2}-99),$$
$$b_{24}=-1400\sqrt{2}+1980,$$
$$b_{25}=0.$$

So that by putting the values of b_{2j} and calculating integrals in Eq. (48) we arrive at

$$H_{2N}(f,y,z,c)=\frac{1}{\pi\sqrt{1-c^2}}[9.775250044c^3-17.9093018933c-0.12046718917c$$
$$+\frac{7.84601809837321c}{(c^2+1)(c^2-0.81)}\left(\frac{10\sqrt{-c^2+1}}{c^2+1}-\frac{10c^2}{\sqrt{-c^2+1}(c^2+1)}-\frac{20c^2\sqrt{-c^2+1}}{(c^2+1)^2}\right)$$
$$\times ln\left|\frac{0.9\sqrt{-c^2+1}-0.43588989435406735522c}{-0.9\sqrt{-c^2+1}-0.43588989435406735522c}\right|\Big]. \quad (63)$$

We have the following tables for $N=\{5,10\}$ to illustrate the numerical results.

N	$c\in(y,z)$	Exact solution	AQS (48), for $r=2$	Absolute error
5	-0.8999	15775.40023417081857	15775.99211049900534	0.59187632818676623
	-0.5961	11.90568247269098462	12.15840661577127353	0.252724143080288913
	-0.0121	0.344893209594347437	0.419431007128060391	0.0745377975337129535
	0.0121	-0.3448932095943474365	-0.213466890190204709	0.1314263194041427275
	0.5961	-11.905682472690984621	-11.988294275144089341	0.082611802453104720
	0.8999	-15775.400234170818575	-15776.882434008911085	1.482199838092510
N	$c\notin(y,z)$	Exact solution	AQS (48), for $r=2$	Absolute error
	-0.901	-15879.854298722082411	-15878.260938214010282	1.593360508072129

5	-0.999	-170.91984259846625914	-169.83529356167509888	1.0845490368116026
	0.901	1618.5985301693003358	1617.1078118275591842	1.4907183417411516
	0.999	170.91984259846625913	169.58311243636870097	1.3367293621755816

Table 4: Results of HSIs (11) with $w_2(t) = \frac{1}{\sqrt{1-t^2}}$ for $N = 5$

N	$c \in (y,z)$	Exact solution	AQS (48), for $r = 2$	Absolute error
10	-0.8999	15775.40023417081857	15775.33014487538023	0.070089295438347
	-0.5961	11.90568247269098462	11.86073534246962517	0.044947130221359448
	-0.0121	0.344893209594347437	0.3435474308856477838	0.0013457787086996528
	0.0121	-0.344893209594347437	-0.3435474308856477838	0.0013457787086996527
	0.5961	-11.905682472690984621	-11.860735342469625173	0.044947130221359448
	0.8999	-15775.400234170818575	-15775.330144875380227	0.070089295438348
N	$c \notin (y,z)$	Exact solution	AQS (48), for $r = 2$	Absolute error
10	-0.901	-1618.5985301693003358	-1618.6686246875732987	0.0700945182729629
	-0.999	-170.91984259846625914	-171.53441737282355402	0.61457477435729488
	0.901	1618.5985301693003358	1618.6686246875732987	0.0700945182729629
	0.999	170.91984259846625913	171.53441737282355402	0.61457477435729489

Table 5: Numerical results of HSIs (11) with $w_2(t) = \frac{1}{\sqrt{1-t^2}}$ for $N = 10$

From the Table 2-5, we can conclude that error of proposed method decreases gradually when number of points are increases. One of the features of the problem is that when the singular point "x" approaches to the end points then values of the HSIs drastically increases fortunately proposed method also approaches to the exact solution very well.

5. Conclusion

In this paper, we have constructed AQS based on Gauss-Chebyshev quadrature rule using truncated series of Chebyshev polynomials of the first and second kind, for the weighted Hadamard type singular integrals on the variable and fixed interval. We have also found the formulas for unknown discrete coefficients by the use of collocation points and composite trapezoidal rule. Numerical examples showed the validity and accuracy of the methods. One of features of the developed method is that error term decreases gradually whenever function is in the form of rational function. For polynomial function approximate solution coincides with the exact solution.

6. Acknowledgement

The authors gratefully acknowledge the support of Research Management Center (RMC), Tashkent Institute of Irrigation and Agricultural Mechanization Engineering, National Research University ("TIIAME", NRU)

References

[1] Bogolubov N.N., Mesheryakov V.A., Tavkhelidze A.N., An application of Muskhelishvili's methods in the theory of elementary particles. Proceedings of Symposium on Solid Mechanics and Related Problems of Analysis, Tbilisi, Vol. 1: (1971), 5–11 (in Russian).

[2] Brown D.D., Jackson A.D., The Nucleon–Nucleon Interaction, North-Holland, Amsterdam.

[3] Ioakimidis N.I., Two methods for the numerical solution of Bueckher's SIEs for plane elasticity crack problems, Comput. Methods Appl. Mech. Eng. 2: (1982), 169–177.

[4] Lifanov I.K., Singular Integral Equations and Discrete Vortices, VSP, Utrecht, The Netherlands, (1996).

[5] Lifanov I.K., Poltavskii L.N., Vainikko G.M., Hypersingular Integral Equations and Their Applications, Chapman Hall/CRC, CRC Press Company, London, New York. (2004).

[6] Boykov I.V., Ventseland E.S., Boykova A.I., Accuracy optimal methods for evaluating hypersingular integrals, J. Appl. Numer. Maths., 59, (2009), 1366–1385.

[7] Zhang X., Wu J., Yu D., Super-convergence of the composite Simpson's rule for a certain finite-part integral and its applications, J. Appl. Numer. Maths., 223, (2009), 598–613.

[8] Hui C.Y., Shia D., Evaluation of hypersingular integrals using Gaussian Quadrature, Int. J. Numer. Meth. Engineering, 44, (1999), 205–214.

[9] Kutt H.R., Quadrature formulae for finite part integrals. Report WISK 178, The National Research Institute for Mathematical Sciences, Pretoria. (1975).

[10] Chen Y.Z.,. A numerical solution technique of hypersingular integral equation for curved cracks, Eng. Frac. Mech., 46(2), (1993), 215–283.

[11] Nik Long N.M.A., Eskhuvatov Z.K.,. Hypersingular integral equation for multiple curved cracks problem in plane elasticity, Int. J. Solids Struct. 46, (2009), 2611–2617.

[12] Chan Y.Sha., Fannjiang A.C., Paulino G.H., Feng B.F., Finite Part Integrals and Hypersingular Kernels, Advances in Dynamical Systems, 14 (2007), 264–269.

[13] Chan Y.Sha, Fannjiang A.C., Paulino G.H. Integral equations with hypersingular kernels—theory and applications to fracture mechanics, International Journal of Engineering Science, 41 (2003), 683–720.

[14] Kim Ph.. A Chebyshev Quadrature Rule for One Sided Finite Parts Integrals, JAT, 111, (2001), 196–219.

[15] Obays S.J., Eshkuvatov Z.K., Nik Long N.M.A., On Error Estimation of Automatic Quadrature Scheme for the Evaluation of Hadamard Integral of Second Order Singularity, U.P.B. Sci. Bull., Series A, 75(1), (2013), 85-98.

[16] Bichi S.L., Eshkuvatov Z.K., Nik Long N.M.A.,. A Note on Evaluating Weighted Hadamard Integral of Second Order Singularity Using Airfoil Polynomials, Uzbek Mat. Journal, 4, (2013), 94–104.

[17] Eskhuvatov Z.K., Ahmedov A., Nik Long N.M.A., Amalina N.J., (2011). Approximating Cauchy-type singular integral by an automatic quadrature scheme. Journal of Computational and Applied Mathematics, 235, (2011), 4675–4686.

[18] Shadimetov Kh.M., Akhmedov D.M. An optimal approximate solution of the Ist kind Fredholm singular integral equations. Filomat Vol. 38:30, (2024), 10765–10796.

[19] Akhmedov D.M., Pirnapasov S.O. Optimal approximation techniques for solving Hadamard-type integral equations, Uzbek Mathematical Journal, Vol. 69, Issue 4, (2025), pp.15-21.

[20] Eshkuvatov Z.K., Mamatova H.X., Ismail Sh., Dereman F., Modified Homotopy Perturbation Method for Solving Singular Integral Equations of the First Kind. Journal of Computational and Applied Mathematics, Vol. 471, (2026), 116753.

[21] Eshkuvatov Z.K., Mamatova H.X., Altynbekov Sh.Y., Kuyliev Sh, Nurillaev M, Highly Accurate Numerical Method for First-Kind Singular Integral Equations, MDPI: Mathematics, 14, (2026), 3405.

[22] Mason J.C. & Handscomb D.C., Chebyshev polynomials, CRC Press LLC. (2003)

[23] Israilov M.I., Semi-Analytic Solution of Singular Integral Equations of First Kind by Quadrature Formula, Inst. Maths. under National University of Uzbekistan (in Russian). 21, (1990).

[24] Hasegawa T., Torii T., An automatic quadrature for Cauchy principal value integrals, Math. Comp. 56, (1991), 741–754.

JMMME
JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING

CONTENTS

00010	**Comparing Stochastic Training Environments for Deep Reinforcement Learning with Deep Q-network: Learning Stability and Transfer to Real Data** Yuriy Yurchenko, Dr. Olexandr Zakovorotnyi (Author)	4–18
00011	**Optimization of Irrigation Systems and Ecological Assessment: A Case Study of Andijan Region, with a Forecast to 2032** Ibrohimjon Abdurahmonov (Author)	19–28
00012	**Evaluation of Cement Production for Continuous and Discrete Flow Systems by Applying a Discrete Event Simulation Approach** Wan Laailatul, Mohd Kamal Nawawi, Nur Anis Hannan Md Shahid (Author)	29–35
00013	**Optimization of Heat Transfer in MHD Mixed Convection Hybrid Nanofluid Flow Over a Shrinking Plate Using Response Surface Methodology** NORIHAN MD ARIFIN, Nor Ain Azeany Mohd Nasir, Nur Fatin Irwani Muhammad Fadzil, Rusya Iryanti Yahya, Nur Syahirah Wahid, Mohd Syazwan Mohamad Anuar (Author)	36–65
00014	**An Automatic Quadrature Scheme for Bounded and Unbounded Weighted Hypersingular Integrals on the Variable Interval** Zainidin Eshkuvatov (Author)	66–80
00015	**A boundary value problem with a conormal derivative and a nonlocal integral condition for a degenerate elliptic equation of the second kind** Akmaljon Abdullaev (Author)	81–97