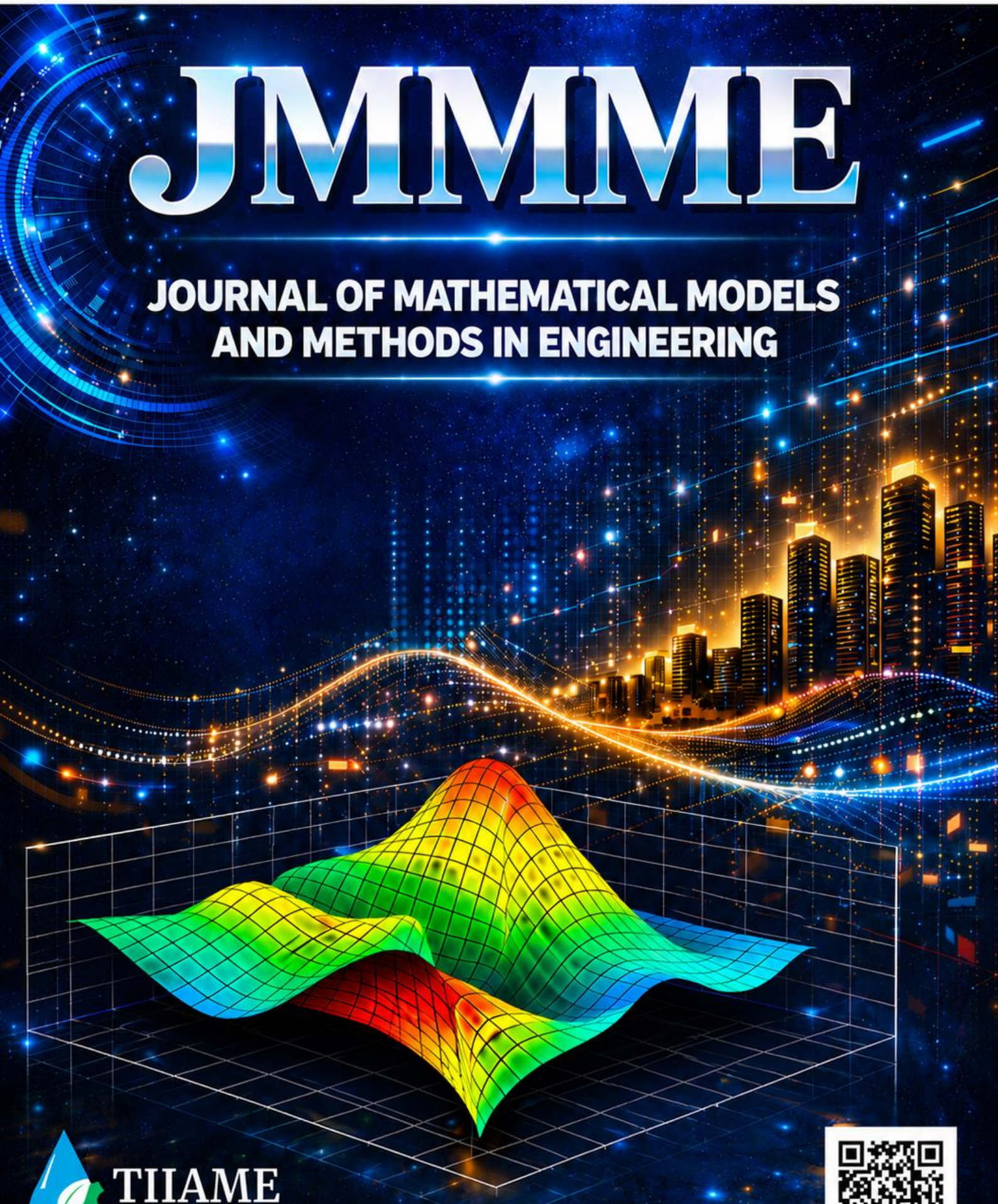
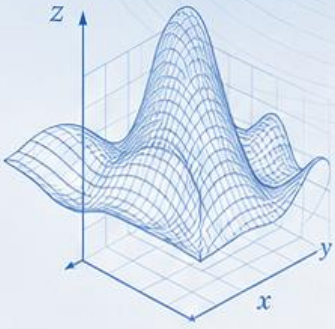
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JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING



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$$\frac{dx}{dt} = Ax + b(t)$$

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u + f$$

$$\sum_{i=1}^{n} a_i x_i + b$$

$$f(x) = \int_a^b f(x)\,dx$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

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$$\sum_{i=1}^{n} w_i x_i$$

The journal specializes in publishing scientific research on mathematical modeling, numerical methods, computational algorithms, optimization, and analytical approaches across various fields of engineering — including civil engineering, mechanical engineering, energy, hydraulic engineering, materials science, robotics, and environmental protection. It serves as a scientific platform for scholars, engineers, educators, and doctoral researchers to exchange theoretical and applied knowledge.



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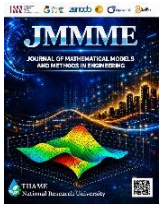
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A boundary value problem with a conormal derivative and a nonlocal integral condition for a degenerate elliptic equation of the second kind

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ABSTRACT

A boundary value problem for a degenerate elliptic equation of the second kind in a finite domain bounded by a smooth curve and a segment of the degeneracy line is investigated. A condition with a conormal derivative, generalizing the Poincaré condition, is specified on the curve, and a nonlocal integral condition containing a normal derivative and negative-order fractional Riemann-Liouville integrals is specified on the degeneracy line. The uniqueness of the solution is established using the energy integral method under the condition that the product of the boundary condition coefficients is nonnegative and the nonlocal condition coefficients have a definite sign. The existence of a solution is proved for a nonzero coefficient of the conormal derivative by reducing to an auxiliary problem with a given trace on the degeneracy line: a Green's function is constructed, and a functional relationship is derived between the trace of the solution and its normal derivative, which, using lemmas on the transformation of fractional integrals, is reduced to a singular integral equation of normal type with a zero index. Using the Carleman-Vekua regularization method, it is reduced to a Fredholm equation of the second kind, the solvability of which follows from the uniqueness of the solution. The results develop the theory of boundary value problems with a conormal derivative for degenerate elliptic equations

1. Introduction

Boundary value problems with the Poincaré condition and with a conormal derivative for the Tricomi, Lavrentiev-Bitsadze, and Gellerstedt equations have been studied in sufficient detail in [1–5]. Significant contributions to the study of such problems were made by L.K. Vostrova, B.V. Melentyev, M.S. Salakhitdinov, Z. Kadyrov, B. Mingziyaev, B. Islomov, and others. However, boundary value problems with a conormal derivative for degenerate elliptic equations of the second kind, complicated by nonlocal integral conditions on the degeneracy line, have not been studied sufficiently.

This paper investigates one such problem. A degenerate elliptic equation of the second kind with degeneracy exponent ranging from minus one to zero is considered in a finite, simply connected

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domain bounded by a smooth curve with endpoints on a degeneracy line and a segment of this line. A boundary condition with a conormal derivative, generalizing the classical Poincaré condition, is specified on the curve, and a nonlocal integral condition containing the normal derivative of the unknown function and negative-order Riemann-Liouville fractional integrals is specified on the degeneracy line. This formulation combines features of problems with a conormal derivative and nonlocal problems with fractional integrals, making it relevant both from a theoretical perspective and in applications to models of transport processes in media with a fractal structure.

The aim of this paper is to establish uniqueness and existence theorems for a solution to the problem under consideration. To prove uniqueness, the energy integral method is applied under the condition that the product of the boundary condition coefficients is non-negative and the non-local condition coefficients have a definite sign; if the coefficient of the unknown function does not vanish, the maximum principle is additionally used. The existence of a solution for a non-zero coefficient of the conormal derivative is proved by reducing the problem to an auxiliary problem with a given trace of the solution on the degeneracy line: the Green's function is constructed using the Gaussian hypergeometric function, a functional relationship is derived between the trace of the solution and its normal derivative, transferred from the domain to the degeneracy line, and, using lemmas on the transformation of fractional integrals, it is reduced to a singular integral equation of normal type with zero index. The latter is reduced by the Carleman-Vekua regularization method to a Fredholm equation of the second kind, the solvability of which follows from the uniqueness of the solution.

The paper consists of three sections. The first presents the problem statement and formulates the conditions on the initial data. The second proves the uniqueness theorem for the solution. The third examines the solvability of the problem: an auxiliary problem with a given trace is considered, its Green's function is constructed, functional relations are derived, and an existence theorem is proved.

2. Statement of the problem

In this paper, we first investigate a boundary value problem with a conormal and integral condition for an elliptic equation of the second kind

$$y^m u_{xx} + u_{yy} = 0, \qquad -1 < m < 0. \tag{1}$$

Let be D a finite single-valued domain in the plane (x, y), bounded by a curve σ at $x > 0,\ y > 0$ with ends at points $A(0,0),\ B(1,0)$ and a segment AB of the axis Ox.

Let's introduce the notation

$$J = \{(x, y):\ 0 < x < 1,\ y = 0\},\ \partial D = \overset{\mathrm{r}}{\sigma} \cup \overline{AB},\quad 2\beta = \frac{m}{m+2},$$

moreover

$$-\frac{1}{2} < \beta < 0. \tag{2}$$

In the domain D for equation (1) we investigate the following problem.

Problem CK**.** It is required to find a function $u(x, y)$ with the following properties:

1) $u(x, y) \in C(\bar{D}) \cup C^1(D \cup \sigma \cup J)$ – u_x and u_y can turn into infinity of order less than -2β at the points $A(0,0)$ and $B(1,0)$;

2) $u(x, y)$ – twice continuously differentiable solution of equation (1) in the domain D;

3) $u(x, y)$ satisfies the boundary conditions

$$\{\delta(s) A_s[u] + \rho(s) u\}\big|_{\sigma} = \varphi(s), \qquad 0 < s < l, \tag{3}$$

$$a_0(x)u_y(x,0)+\sum_{j=1}^{n}a_j(x)D_{0x}^{\alpha_j}u(x,0)+a_{n+1}(x)u(x,0)=b(x),\qquad 0<x<1, \tag{4}$$

where $\delta(s)$, $\rho(s)$, $\varphi(s)$, $a_j(x)$, $b(x)$ are given functions, and

$$b(0)=0,\ \ a_0(1)\neq 0, \tag{5}$$

$$a_0(x)\neq 0,\quad \forall x\in\overline{J}, \tag{6}$$

$$\delta^2(s)+\rho^2(s)\neq 0,\quad \forall s\in[0,l], \tag{7}$$

$$\sum_{j=0}^{n+1}a_j^2(x)\neq 0,\quad \forall(x,0)\in\overline{J}, \tag{8}$$

$$\delta(s),\rho(s),\varphi(s)\in C[0,l], \tag{9}$$

$$a_j(x),\ b(x)\in C\left(\overline{J}\right)\cap C^2(J),\ (j=\overline{0,n+1}), \tag{10}$$

and

$$A_s[u]=y^m\frac{dy}{ds}\frac{\partial u}{\partial x}-\frac{dx}{ds}\frac{\partial u}{\partial y},$$

$\frac{dx}{ds}=-\cos(n,y)$, $\frac{dy}{ds}=\cos(n,x)$, n – the external normal to the curve σ, l – the length of the entire curve σ, s – the length of the arc of the curve σ, measured from the point $B(1,0)$, and $D_{0x}^{\alpha_j}[*]$ is the operator (in the Riemann-Liouville sense) of fractional integration $\alpha_j\in(-1,0)$ given by the formula [10]:

$$D_{0x}^{\alpha_j}f(x)=\frac{1}{\Gamma(-\alpha_j)}\int_0^x\frac{f(t)dt}{(x-t)^{\alpha_j+1}},\qquad -1<\alpha_j<0. \tag{11}$$

Let it $\alpha=\max\limits_{1\le j\le n}\{|\alpha_j|\}$, be so

$$\alpha\le -2\beta. \tag{12}$$

We will assume that the curve σ satisfies the following condition:

1) functions $x(s)$, $y(s)$ that give a parametric equation of the curve σ have continuous derivatives $x'(s),y'(s)$ that do not simultaneously vanish and have second derivatives that satisfy the Hölder condition of order $\kappa\ (0<\kappa<1)$ in the interval $0\le s\le l$;

2) in the neighborhood of the end points the curve σ satisfies the inequalities:

$$\left|\frac{dx}{ds}\right|\le const\ y^{m+1}(s), \tag{13}$$

moreover $x(l)=y(0)=0,\quad x(0)=1,\quad y(l)=0.$

3. Uniqueness of the solution to the problem CK

Let

$$-1<\alpha_j<0,\ \ \delta(s)\neq 0,\ \ \forall s\in[0,l], \tag{14}$$

then the uniqueness of the solution to the problem CK is proven by the method of energy integrals and the following holds:

Theorem 1. If conditions (2), (5), (6), (7), (8) are satisfied and

$$\delta(s)\rho(s)\geq 0,\quad 0\leq s\leq l, \tag{15}$$

$$\frac{a_{n+1}(x)}{a_0(x)}\leq 0, \tag{16}$$

$$\left(\frac{a_j(x)}{a_0(x)}\right)'_x\geq 0,\quad \frac{a_j(1)}{a_0(1)}\leq 0,\quad \left(j=\overline{1,n}\right), \tag{17}$$

then the problem CK in the domain D cannot have more than one solution.

Proof of Theorem 1. Let $\varphi(s)\equiv b(x)\equiv 0$. Consider the identity

$$u\left[y^m u_{xx}+u_{yy}\right]=\frac{\partial}{\partial x}\left[y^m u\,u_x\right]+\frac{\partial}{\partial y}\left[u\,u_y\right]-y^m u_x^2-u_y^2. \tag{18}$$

Integrating (18) over the region D,

$$0=\iint_D u\left[y^m u_{xx}+u_{yy}\right]dxdy=\iint_D\left\{\frac{\partial}{\partial x}\left[y^m uu_x\right]+\frac{\partial}{\partial y}\left[uu_y\right]\right\}dxdy-\iint_D\left[y^m u_x^2+u_y^2\right]dxdy,$$

and then applying the Gauss-Ostrogradsky formula [10]

$$\iint_D\left\{\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right\}dxdy=\int_{\partial D}Qdx+Pdy$$

we will receive

$$0=\iint_D u\left[y^m u_{xx}+u_{yy}\right]dxdy=-\iint_D\left[y^m u_x^2+u_y^2\right]dxdy+\int_{\partial D}u\left[y^m u_x dy-u_y dx\right].$$

Hence, taking into account that $AB\colon y=0\Rightarrow dy=0$, and $dy=\cos(n,x)ds$, $dx=-\cos(n,y)ds$ we have,

$$=-\iint_D\left[y^m u_x^2+u_y^2\right]dxdy-\int_0^1\tau(x)\nu(x)dx+\int_\sigma u\,A_s[u]ds, \tag{19}$$

where

$$u(x,0)=\tau(x),\ (x,0)\in\overline{J},\qquad u_y(x,0)=\nu(x),\quad (x,0)\in J. \tag{20}$$

Since $\varphi(s)\equiv 0$, then from (19) taking into account (3) we $\delta(s)\neq 0$ find:

$$\iint_D\left[y^m u_x^2+u_y^2\right]dxdy+\int_0^1\tau(x)\nu(x)dx+\int_\sigma\frac{\delta(s)\rho(s)}{\delta^2(s)}u^2ds=0. \tag{21}$$

By virtue of (15), from the third integral of equality (21) we have

$$R_3=\int_\sigma\frac{\delta(s)\rho(s)}{\delta^2(s)}u^2ds\geq 0. \tag{22}$$

Now we will show that the second term on the left side of equality (21) is non-negative.
By virtue of (8) from (4) taking into account (20) we have

$$\nu(x)=-\left[\sum_{j=1}^{n}\frac{a_j(x)}{a_0(x)}D_{0x}^{\alpha_j}\tau(x)+\frac{a_{n+1}(x)}{a_0(x)}\tau(x)\right].$$

From here, by virtue of (12), from the second term of equality (21) we obtain

$$R_2=\int_0^1\tau(x)\nu(x)dx=-\int_0^1\sum_{j=1}^{n}\frac{a_j(x)\tau(x)}{\Gamma\left(-\alpha_j\right)a_0(x)}dx\int_0^x\frac{\tau(t)dt}{\left(x-t\right)^{1+\alpha_j}}-$$

$$-\int_0^1\frac{a_{n+1}(x)}{a_0(x)}\tau^2(x)dx=R_{21}+R_{22}\qquad(23)$$

Using formula [1]

$$\left|x-t\right|^{-\gamma}=\frac{1}{\Gamma(\gamma)\cos\dfrac{\pi\gamma}{2}}\int_0^\infty z^{\gamma-1}\cos[z(x-t)]dz,\qquad 0<\gamma<1$$

from (23) we find

$$R_{21}=-\int_0^1\sum_{j=1}^{n}\frac{a_j(x)\tau(x)dx}{a_0(x)\Gamma\left(-\alpha_j\right)\Gamma(1+\alpha_j)\cos\pi(1+\alpha_j)/2}\int_0^x\tau(t)dt\int_0^\infty z^{\alpha_j}\cos z(x-t)dz=$$

$$=-2\sum_{j=1}^{n}\cos\frac{\pi\alpha_j}{2}\int_0^\infty z^{\alpha_j}dz\int_0^1\frac{a_j(x)\tau(x)}{a_0(x)}dx\int_0^x\left[\cos zx\cos zt+\sin zx\sin zt\right]\tau(t)dt=$$

$$=-\sum_{j=1}^{n}\cos\frac{\pi\alpha_j}{2}\int_0^\infty z^{\alpha_j}dz\int_0^1\frac{a_j(x)}{a_0(x)}\cdot\frac{\partial}{\partial x}\left[\left(\int_0^x\tau(t)\cos ztdt\right)^2+\left(\int_0^x\tau(t)\sin ztdt\right)^2\right].$$

Integrating by parts the integrals over x, we have

$$R_{21}=-\sum_{j=1}^{n}\cos\frac{\pi\alpha_j}{2}\int_0^\infty z^{\alpha_j}dz\left\{\frac{a_j(x)}{a_0(x)}\left[\left(\int_0^x\tau(t)\cos ztdt\right)^2+\left(\int_0^x\tau(t)\sin ztdt\right)^2\right]\right|_{x=0}^{x=1}-$$

$$-\int_0^1\left(\frac{a_j(x)}{a_0(x)}\right)_x'\left[\left(\int_0^x\tau(t)\cos ztdt\right)^2+\left(\int_0^x\tau(t)\sin ztdt\right)^2\right]dx\Bigg\}=$$

$$=-\sum_{j=1}^{n}\cos\frac{\pi\alpha_j}{2}\int_0^\infty z^{\alpha_j}\frac{a_j(1)}{a_0(1)}\left[\left(\int_0^x\tau(t)\cos ztdt\right)^2+\left(\int_0^x\tau(t)\sin ztdt\right)^2\right]dz+$$

$$+\sum_{j=1}^{n}\cos\frac{\pi\alpha_j}{2}\int_0^\infty z^{\alpha_j}dz\int_0^1\left(\frac{a_j(x)}{a_0(x)}\right)_x'\left[\left(\int_0^x\tau(t)\cos ztdt\right)^2+\left(\int_0^x\tau(t)\sin ztdt\right)^2\right]dx.$$

From here, by virtue of (17) we have

$$R_{21}\geq 0.\qquad(24)$$

Taking into account (16) from (23) we have

$$R_{22}\geq 0.\qquad(25)$$

By virtue of (24) and (25) from (23) we obtain

$$R_2 \geq 0. \tag{26}$$

Teaches (22) and (26) from (21) it follows that $u_x = u_y = 0$ in D, that is, $u = const$ in $(x,y) \in D$, also from the consequence of the vanishing of each term in (21), we have and $u = 0$ in $\bar{\sigma}$. Consequently, $u \equiv 0$ in $\bar{D}$ when $\delta(s) \neq 0$.

Note: Uniqueness of the solution to the problem CK is $\rho(s) \neq 0,\ \forall\ s \in [0,l]$ proved using the maximum principle[7].

Theorem 1 is proven.

4. Existence of a solution to the problem CK when $\delta(s) \neq 0$

Consider the following auxiliary problem.

Problem DK**:** Find a solution $u(x,y) \in C(\bar{D}) \cap C^1(D \cup \sigma \cup J) \cap$ in the area $D \cap C^2(D)$ equation (1), satisfying conditions (3) and

$$u\big|_{y=0} = \tau(x), \qquad 0 \leq x \leq 1, \tag{27}$$

where $\tau(x) -$ is a given continuous function, and $\tau(x)$ satisfies the Hölder condition with an exponent $\gamma_0 \geq 1 - 2\beta$ in the interval $(0,1)$ and can be represented in the form

$$\tau(x) = \int_x^1 (t-x)^{-2\beta} T(t)dt, \tag{28}$$

where the function $T(t)$ is continuous in (0,1) and integrable in $[0,1]$.

The uniqueness of the solution to the problem DK follows from identity (21).

The solution of the problem DK with conditions (3) and (27) for equation (1) in the region D exists, is unique and can be represented in the form [10. see (10.78)]:

$$u(x,y) = \int_0^1 \tau(\xi) \frac{\partial}{\partial \eta} G_2(\xi, 0; x, y)d\xi + \int_0^l \frac{\varphi(s)}{\delta(s)} G_2(\xi, \eta; x, y)ds, \tag{29}$$

where is $G_2(\xi,\eta;x,y) -$ the Green's function of the problem DK for equation (1) and it has the form [10]:

$$G_2(\xi,\eta;x,y) = G_{02}(\xi,\eta;x,y) + H_2(\xi,\eta;x,y), \tag{30}$$

here is $G_{02}(\xi,\eta;x,y) -$ the Green's function of the problem DK for equation (1) for normal region D_0 bounded by a segment $\overline{AB}$ and a normal curve $\sigma_0 : \left(x - \frac{1}{2}\right)^2 + \frac{4}{(m+2)^2} y^{m+2} = \frac{1}{4}$,

$$H_2(\xi,\eta;x,y) = G_2(\xi,\eta;x,y) - G_{02}(\xi,\eta;x,y) =$$

$$= \int_0^l \lambda_2(s;\xi,\eta) \left\{ A_s \left[G_{02}(\xi(s),\eta(s);x,y) \right] + \frac{\rho(s)}{\delta(s)} G_{02}(\xi(s),\eta(s);x,y) \right\} ds, \tag{31}$$

where is $\lambda_2(s;\xi,\eta) -$ the solution of the integral equation

$$\lambda_2(s;\xi,\eta)+2\int_0^l \lambda_2(t;\xi,\eta)\Big\{A_s\Big[q_2\big(\xi(t),\eta(t);x(s),y(s)\big)\Big]+$$

$$+\frac{\rho(s)}{\delta(s)}q_2\big(\xi(t),\eta(t);x(s),y(s)\big)\Big\}dt=-2q_2(\xi(s),\eta(s);\xi,\eta), \qquad (32)$$

a $q_2(x,y,x_0,y_0)$ - the fundamental solution of equation (1) and it has the form:

$$q_2(x,y,x_0,y_0)=k_2\left(\frac{4}{m+2}\right)^{4\beta-2}\left(r_1^2\right)^{-\beta}(1-\sigma)^{1-2\beta}F(1-\beta,1-\beta,2-2\beta;1-\sigma) \qquad (33)$$

where

$$\left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\}=(x-x_0)^2+\frac{4}{(m+2)^2}\left(y^{\frac{m+2}{2}}\ \mathrm{m}y_0^{\frac{m+2}{2}}\right)^2,$$

$$\sigma=\frac{r^2}{r_1^2}, \qquad \beta=\frac{m}{2(m+2)}<0, \qquad k_2=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)},$$

and $F(a,b,c;z)$ – the Gaussian hypergeometric function[7].

Differentiating by y equation (29), then tending y to zero taking into account (30), (33) we obtain the functional relationship between $\tau(x)$ and $\nu(x)$, transferred from the region D to J:

$$\nu(x)=k_2\int_0^1 |t-x|^{2\beta-2}\tau(t)dt-k_2\int_0^1\frac{\tau(t)dt}{(t+x-2tx)^{2-2\beta}}+$$

$$+\int_0^1\tau(t)\frac{\partial^2 H_2(t,0;x,0)}{\partial\eta\partial y}dt+\int_0^l\chi(s)\frac{\partial q_2(t,\eta;x,0)}{\partial y}ds, \qquad (34)$$

where $\chi(s)$ is the solution to the integral equation

$$\chi(s)+2\int_0^l\chi(t)\Big\{A_s\Big[q_2\big(\xi(t),\eta(t);x(s),y(s)\big)\Big]+$$

$$+\frac{\rho(s)}{\delta(s)}q_2\big(\xi(t),\eta(t);x(s),y(s)\big)\Big\}dt=\frac{2\varphi(s)}{\delta(s)}. \qquad (35)$$

The following lemmas hold.

Lemma 1. Let $\tau(x)$ a function belong to the class $C^{(1,\gamma_1)}(0,1)$, $\gamma_1\ge -2\beta$, then the following $(0,1)$ identities hold true $(0<-2\beta<1)$:

$$\int_0^x(x-t)^{2\beta-2}\tau(t)dt=\frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_0^x(x-t)^{2\beta}\tau(t)dt, \qquad (36_1)$$

$$\int_x^1(t-x)^{2\beta-2}\tau(t)dt=\frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_x^1(t-x)^{2\beta}\tau(t)dt. \qquad (36_2)$$

Proof of Lemma 1. We will put the identities (36_1) and (36_2) respectively in the form

$$T_1(x)=\lim_{\varepsilon\to 0}T_{1\varepsilon}(x)=\lim_{\varepsilon\to 0}\left\{\frac{d^2}{dx^2}\int_0^{x-\varepsilon}\left(x-t\right)^{2\beta}\tau(t)dt\right\}$$

and

$$T_2(x)=\lim_{\varepsilon\to 0}T_{2\varepsilon}(x)=\lim_{\varepsilon\to 0}\left\{\frac{d^2}{dx^2}\int_{x+\varepsilon}^{1}\left(t-x\right)^{2\beta}\tau(t)dt\right\}.$$

From this we have

$$T_{1\varepsilon}(x)=\frac{d}{dx}\left[\varepsilon^{2\beta}\tau(x-\varepsilon)+2\beta\int_0^{x-\varepsilon}\left(x-t\right)^{2\beta-1}\tau(t)dt\right]=$$

$$=\varepsilon^{2\beta}\tau'(x-\varepsilon)+2\beta\varepsilon^{2\beta-1}\tau(x-\varepsilon)+2\beta(2\beta-1)\int_0^{x-\varepsilon}\left(x-t\right)^{2\beta-2}\tau(t)dt,\qquad(37_1)$$

$$T_{2\varepsilon}=\frac{d}{dx}\left[-\varepsilon^{2\beta}\tau(x+\varepsilon)-2\beta\int_{x+\varepsilon}^{1}(t-x)^{2\beta-1}\tau(t)dt\right]=$$

$$=-\varepsilon^{2\beta}\tau'(x+\varepsilon)+2\beta\varepsilon^{2\beta-1}\tau(x+\varepsilon)+2\beta(2\beta-1)\int_{x+\varepsilon}^{1}\left(t-x\right)^{2\beta-2}\tau(t)dt.\qquad(37_2)$$

Passing to the limit in (37_j) with $\varepsilon\to 0$ taking into account the conditions of Lemma 1, we obtain the validity of identity (36_j) $(j=1,2)$.

Lemma 1 is proven.

Lemma 2. Let $\tau(x)\in C^{(1,\gamma_1)}(0,1)$, $\gamma_1\geq -2\beta$ and be representable in the form (28), then the following $(0,1)$ identities hold $(0<-2\beta<1)$:

$$\int_0^x\left(x-t\right)^{2\beta-2}\tau(t)dt=\frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_0^x\left(x-t\right)^{2\beta}\tau(t)dt=$$

$$=\frac{\Gamma(1+2\beta)\Gamma(1-2\beta)}{2\beta(2\beta-1)}D_{0x}^{1-2\beta}D_{x1}^{2\beta-1}T(x)=$$

$$=\frac{\pi\operatorname{ctg}\pi 2\beta}{1-2\beta}T(x)-\frac{1}{1-2\beta}\int_0^1\left(\frac{t}{x}\right)^{1-2\beta}\frac{T(t)}{t-x}dt,\qquad(38)$$

$$\int_x^1\left(t-x\right)^{2\beta-2}\tau(t)dt=\frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_x^1\left(t-x\right)^{2\beta}\tau(t)dt=$$

$$=\frac{\Gamma(1+2\beta)\Gamma(1-2\beta)}{2\beta(2\beta-1)}D_{x1}^{1-2\beta}D_{x1}^{2\beta-1}T(x)=\frac{\pi}{(2\beta-1)\sin 2\beta\pi}T(x),\qquad(39)$$

$$\int_0^1\left(t+x-2tx\right)^{2\beta-2}\tau(t)dt=\int_0^1\left(t+x-2tx\right)^{2\beta-2}dt\int_t^1\left(z-t\right)^{-2\beta}T(z)dz=$$

$$= \frac{1}{1-2\beta}\int_0^1 \left(\frac{t}{x}\right)^{1-2\beta} \frac{T(t)dt}{x+t-2xt}. \qquad (40)$$

Proof of Lemma 2. Using the definitions of the fractional order integro-differentiation operator [10. § 4, see. (4.1), (4.6)]:

$$D_{px}^{\sigma} f(x) = \begin{cases} \dfrac{1}{\Gamma(-\sigma)}\displaystyle\int_p^x (x-t)^{-\sigma-1} f(t)dt, & x\in(p,q), \quad \sigma<0, \\ f(x) & \sigma=0, \\ \dfrac{d^n}{dx^n}\left[D_{px}^{\sigma-n} f(x)\right] & n-1<\sigma\le n, \quad n\in N, \end{cases}$$

taking into account (28) and $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$ from the right-hand side of (38) we obtain

$$A_1 = \int_0^x (x-t)^{2\beta-2}\tau(t)dt = \frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_0^x (x-t)^{2\beta}\tau(t)dt =$$

$$= \frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_0^x (x-t)^{2\beta}dt\int_t^1 (z-t)^{-2\beta}T(z)dz =$$

$$= \frac{\Gamma(1+2\beta)\Gamma(1-2\beta)}{2\beta(2\beta-1)}\frac{d^2}{dx^2}D_{0x}^{-(1+2\beta)}D_{x1}^{2\beta-1}T(x) = \frac{\pi}{(2\beta-1)\sin 2\beta\pi}D_{0x}^{1-2\beta}D_{x1}^{2\beta-1}T(x). \qquad (41)$$

By virtue of the formula [10, p. 24, Lemma 4.5]:

$$D_{ax}^{\alpha}D_{xb}^{-\alpha}\Phi(x) = \cos\pi\alpha\,\Phi(x) + \frac{\sin\pi\alpha}{\pi}\int_a^b \left(\frac{t-a}{x-a}\right)^{\alpha}\frac{\Phi(t)}{t-x}dt, \quad 0<\alpha<1$$

from (41) we have

$$A_1 = \frac{\pi\,\mathrm{ctg}\,2\beta\pi}{1-2\beta}T(x) - \frac{1}{1-2\beta}\int_0^1 \left(\frac{t}{x}\right)^{1-2\beta}\frac{T(t)}{t-x}dt.$$

From this follows the validity of identity (38).

Now we will prove the validity of identity (39). By virtue of (28), from the right-hand side of (39) we obtain

$$A_2 = \int_x^1 (t-x)^{2\beta-2}\tau(t)dt = \frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_x^1 (t-x)^{2\beta}\tau(t)dt =$$

$$= \frac{1}{2\beta(2\beta-1)}\frac{d^2}{dx^2}\int_x^1 (t-x)^{2\beta}\tau(t)dt\int_t^1 (z-t)^{-2\beta}T(z)dz.$$

Next, using the definitions of the fractional order integro-differentiation operator [10. § 4, см. (4.13), (4.14)]:

$$D_{xq}^{\sigma}f(x)=\begin{cases}\dfrac{1}{\Gamma(-\sigma)}\displaystyle\int_x^b(t-x)^{-\sigma-1}f(t)dt, & x\in(p,q), \quad \sigma<0,\\ f(x), & \sigma=0,\\ (-1)^n\dfrac{d^n}{dx^n}\left[D_{xb}^{-(n-\sigma)}f(x)\right], & n-1<\sigma\le n,\ n\in N,\end{cases} \tag{42}$$

and taking into account the formula $D_{xb}^{\alpha}D_{xb}^{-\alpha}f(x)=f(x)$, we have

$$A_2=\frac{\pi}{(2\beta-1)\sin 2\beta\pi}D_{x1}^{1-2\beta}D_{x1}^{-(1-2\beta)}T(x)=\frac{\pi}{(2\beta-1)\sin 2\beta\pi}T(x).$$

From this follows the validity of identity (39).

Let's consider the expression

$$A_3=\int_0^1\left(t+x-2tx\right)^{2\beta-2}\tau(t)dt. \tag{43}$$

Substituting (28) into (40), and then changing the order of integration, we have

$$A_3=\int_0^1\left(t+x-2tx\right)^{2\beta-2}dt\int_t^1\left(z-t\right)^{-2\beta}T(z)dz=\int_0^1T(z)dz\int_0^z\left(z-t\right)^{-2\beta}\left(t+x-2tx\right)^{2\beta-2}dt.$$

Assuming in the inner integral $t=z(1-s)$ and using the formula [10, p. 8, (2.10)]

$$\int_0^1 t^{a-1}\left(1-t\right)^{c-a-1}\left(1-zt\right)^{-b}dt=\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}F\left(a,b,c;z\right),$$

$$0<\mathrm{Re}\,a<\mathrm{Re}\,c,\ \left|\arg(1-z)\right|<\pi, \tag{44}$$

we will receive

$$A_3=\int_0^1T(z)\,z^{-2\beta+1}\left(z+x-2xz\right)^{2\beta-2}dz\int_0^1 s^{-2\beta}\left[1-\frac{(1-2x)zs}{(z+x-2xz)}\right]^{2\beta-2}ds=$$

$$=\frac{\Gamma(1-2\beta)\Gamma(1)}{\Gamma(2-2\beta)}\int_0^1\left(\frac{z}{x+z-2xz}\right)^{1-2\beta}\frac{T(z)}{x+z-2xz}\times$$

$$\times F\left(1-2\beta,2-2\beta,2-2\beta;\frac{(1-2x)z}{x+z-2xz}\right)dz.$$

Taking the formulas $F\left(a,b,b;z\right)=\left(1-z\right)^{-a}$ and $\Gamma(1+a)=a\Gamma(a)$, from the last expression we have

$$A_3=\frac{1}{1-2\beta}\int_0^1\left(\frac{t}{x}\right)^{1-2\beta}\frac{T(t)dt}{x+t-2xt}.$$

From this follows the validity of identity (40).

Lemma 2 is proven.

Thus, substituting (38), (39), (40) and (28) into (34) we obtain the functional relationship between $T(x)$ and $\nu(x)$, transferred from the region D to J:

$$\nu(x)=-\frac{k_2\pi tg\beta\pi}{1-2\beta}T(x)+\frac{k_2}{1-2\beta}\int_0^1\left(\frac{t}{x}\right)^{1-2\beta}T(t)\left[\frac{1}{x-t}-\frac{1}{x+t-2xt}\right]dt+$$

$$+\int_0^1 T(t)dt\int_0^t\frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\,\partial y}(t-z)^{-2\beta}\,dz+\int_0^l\frac{\partial q_2(t,\eta;x,0)}{\partial y}\chi(s)ds,\quad (x,0)\in J\,. \quad (45)$$

By virtue of (20), taking into account (8), (11), (28), from (4) on the interval J we obtain a functional relationship between $\tau(x)$ and $\nu(x)$:

$$\nu(x)=-\sum_{j=1}^{n}\frac{a_j(x)}{a_0(x)}\cdot\frac{1}{\Gamma(-\alpha_j)}\int_0^x(x-t)^{-\alpha_j-1}dt\int_t^1(z-t)^{-2\beta}T(z)dz-$$

$$-\frac{a_{n+1}(x)}{a_0(x)}\int_x^1(t-x)^{-2\beta}T(t)dt+\frac{b(x)}{a_0(x)},\qquad (x,0)\in J. \quad (46)$$

Theorem 2. If conditions (2), (5)-(10) and (12)-(14) are satisfied, then in the region D the solution of the problem CK exists.

Proof of Theorem 2. Eliminating $\nu(x)$ from relations (45) and (46) we have

$$T(x)-\frac{ctg\beta\pi}{\pi}\int_0^1\left(\frac{t}{x}\right)^{1-2\beta}\left[\frac{1}{x-t}-\frac{1}{x+t-2xt}\right]T(t)dt-$$

$$-\frac{(1-2\beta)ctg\beta\pi}{k_2\pi}\int_0^1 T(t)dt\int_0^t(t-z)^{-2\beta}\frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\partial y}dz-$$

$$-\frac{(1-2\beta)ctg\beta\pi}{k_2\pi}\sum_{j=1}^{n}\frac{a_j(x)}{a_0(x)}\cdot\frac{1}{\Gamma(-\alpha_j)}\int_0^x T(t)dt\int_0^t(x-z)^{-\alpha_j-1}(t-z)^{-2\beta}dz-$$

$$-\frac{(1-2\beta)ctg\beta\pi}{k_2\pi}\sum_{j=1}^{n}\frac{a_j(x)}{a_0(x)}\cdot\frac{1}{\Gamma(-\alpha_j)}\int_x^1 T(t)dt\int_0^x(x-z)^{-\alpha_j-1}(t-z)^{-2\beta}dz-$$

$$-\frac{(1-2\beta)ctg\beta\pi}{k_2\pi}\cdot\frac{a_{n+1}(x)}{a_0(x)}\int_x^1(t-x)^{-2\beta}T(t)dt=\frac{(2\beta-1)ctg\beta\pi}{k_2\pi}\cdot\frac{b(x)}{a_0(x)}+$$

$$+\frac{(1-2\beta)ctg\beta\pi}{k_2\pi}\int_0^l\chi(s)\frac{\partial q_2(\xi(s),\eta(s);x,0)}{\partial y}ds\,,$$

or

$$\overset{\beta}{T}(x)-\gamma_3\int_0^1\left[\frac{1}{x-t}-\frac{1}{x+t-2xt}\right]\overset{\beta}{T}(t)dt-\int_0^1 K(x,t)\overset{\beta}{T}(t)dt=F(x), \quad (47)$$

where $\gamma_3=\frac{1}{\pi}ctg\beta\pi$, $\overset{\beta}{T}(x)=x^{1-2\beta}T(x)$,

$$K(x,t)=\begin{cases} K_1(x,t), & 0\le t\le 1,\\ K_2(x,t), & 0\le t\le x,\\ K_3(x,t)+K_4(x,t), & x\le t\le 1, \end{cases} \tag{48}$$

$$K_1(x,t)=\frac{(1-2\beta)\cdot\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\int_0^t (t-z)^{-2\beta}\cdot\frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\partial y}dz, \tag{49}$$

$$K_2(x,t)=\frac{(1-2\beta)\cdot\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\sum_{j=1}^{n}\frac{a_j(x)}{a_0(x)}\frac{1}{\Gamma(-\alpha_j)}\int_0^t (x-z)^{-\alpha_j-1}(t-z)^{-2\beta}dz, \tag{50}$$

$$K_3(x,t)=\frac{(1-2\beta)\cdot\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\sum_{j=1}^{n}\frac{a_j(x)}{a_0(x)}\frac{1}{\Gamma(-\alpha_j)}\int_0^x (x-z)^{-\alpha_j-1}(t-z)^{-2\beta}dz, \tag{51}$$

$$K_4(x,t)=\frac{(1-2\beta)\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\frac{a_{n+1}(x)}{a_0(x)}(t-x)^{-2\beta}, \tag{52}$$

$$F(x)=\frac{(2\beta-1)\gamma_3 x^{1-2\beta}}{k_2}\left[\frac{b(x)}{a_0(x)}-\int_0^l \chi(s)\frac{\partial q_2(\xi(s),\eta(s);x,0)}{\partial y}ds\right]. \tag{53}$$

Let us examine the kernels and the right-hand side of the singular integral equation (47). It takes place [10, p.181]:

Lemma 3. Let, then $0<x<1,\quad 0<z<1$ the inequality holds

$$\left|\frac{\partial^2 H_2(z,0;x,0)}{\partial\eta\partial y}\right|<C_1(x+z-2xz)^{2\beta-1}. \tag{54}$$

where is C_1 – a constant depending only on the region D.

The proof of Lemma 3 is similar to the proof of Lemma 18.1 [10, pp. 133-136].

By virtue of (54) from (49) we have

$$|K_1(x,t)|\le C_1\frac{(1-2\beta)\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\left|\int_0^t (t-z)^{-2\beta}(x+z-2xz)^{2\beta-1}dz\right|. \tag{55}$$

By replacing variables $z=t(1-\sigma)$ and using formulas (44) from (55) we obtain

$$|K_1(x,t)|\le C_1\frac{(1-2\beta)\gamma_3}{k_2}\left(\frac{x}{x+t-2xt}\right)^{1-2\beta}\int_0^1 \sigma^{-2\beta}\left[1-\frac{(1-2x)t}{x+t-2xt}\sigma\right]^{2\beta-1}d\sigma. \tag{55}$$

By virtue of (44) from (55) we have

$$|K_1(x,t)|\le C_1\frac{(1-2\beta)\gamma_3}{k_2}\left(\frac{x}{x+t-2xt}\right)^{1-2\beta}F\left(1-2\beta,1-2\beta,2-2\beta;\frac{t(1-2x)}{x+t-2xt}\right) \tag{56}$$

Since $c-a-b=2-2\beta-2+4\beta=2\beta<0$, then using formulas [7]:

$$F(a,b,c,z)=(1-z)^{c-a-b}F(c-a,c-b,c;z),\qquad |\arg(1-z)|<\pi \tag{57}$$

and assessments

$$F(a,b,c,z)\le\begin{cases}const & at \quad c-a-b>0, \quad 0\le z\le 1,\\ const(1-z)^{c-a-b} & at \quad c-a-b<0, \quad 0<z<1,\\ const\left[1+ln(1-z)\right] & at \quad c-a-b=0, \quad 0<z<1\end{cases} \tag{58}$$

from (56) we obtain the estimate

$$\left|K_1(x,t)\right|\le C_1C_2\frac{(1-2\beta)\gamma_3}{k_2}\left(\frac{x}{x+t-2xt}\right)^{1-2\beta}\left(\frac{x}{x+t-2xt}\right)^{2\beta}\le\frac{C_3x}{x+t-2xt}. \tag{59}$$

Let's study the nuclei (50). In the integral (50), we make a substitution $z=t\mu$ and from (50), taking into account (44), we obtain

$$K_2(x,t)=\frac{(1-2\beta)\cdot\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\sum_{j=1}^{n}\frac{a_j(x)}{a_0(x)}\frac{x^{-\alpha_j-1}t^{1-2\beta}}{\Gamma(-\alpha_j)}\int_0^1(1-s)^{-2\beta}\left(1-\frac{t}{x}s\right)^{-\alpha_j-1}ds=$$

$$=\frac{(1-2\beta)\cdot\gamma_3}{k_2}\frac{\Gamma(1-2\beta)}{\Gamma(2-2\beta)}\left(\frac{x}{t}\right)^{1-2\beta}\sum_{j=1}^{n}\frac{a_j(x)}{a_0(x)}\frac{x^{-\alpha_j-1}t^{1-2\beta}}{\Gamma(-\alpha_j)}F\left(1,1+\alpha_j,2-2\beta;\frac{t}{x}\right)=$$

$$=\frac{\gamma_3}{k_2}\sum_{j=1}^{n}\frac{a_j(x)}{a_0(x)}\frac{x^{-\alpha_j-2\beta}}{\Gamma(-\alpha_j)}F\left(1,1+\alpha_j,2-2\beta;\frac{t}{x}\right). \tag{60}$$

By virtue of (6), (10), (12) and (58) taking into account $c-a-b=2-2\beta-1-1-\alpha_j=$ $=-2\beta-\alpha_j\ge-2\beta-\alpha\ge0, \quad 0\le t\le x\le1$ from (60) we have the estimate

$$\left|K_2(x,t)\right|\le\frac{\gamma_3}{k_2}\sum_{j=1}^{n}\frac{C_4C_6}{C_5}\frac{x^{-\alpha_j-2\beta}}{\Gamma(-\alpha_j)}\le\frac{\gamma_3C_4C_6}{k_2C_5}x^{-2\beta}\sum_{j=1}^{n}\frac{1}{x^{\alpha_j}\Gamma(-\alpha_j)}\le C_7\,x^{-\alpha-2\beta},$$

or

$$\left|K_2(x,t)\right|\le C_7\,x^{-(\alpha+2\beta)}\le C_8 \quad . \tag{61}$$

We estimate exactly as above $K_3(x,t)$. Indeed, taking into account (6), (10), (12) and (58) $c-a-b=1-\alpha_j-1-2\beta=-2\beta-\alpha_j \quad 0\le x\le t\le1$ from (51) we have

$$\left|K_3(x,t)\right|\le\frac{\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\sum_{j=1}^{n}\frac{C_4C_9}{C_5}\frac{x^{-\alpha_j}t^{-2\beta}}{\Gamma(-\alpha_j)}\le\frac{\gamma_3C_4C_9}{k_2C_5}\frac{x}{t}\times$$

$$\times x^{-2\beta}\sum_{j=1}^{n}\frac{1}{x^{\alpha_j}\Gamma(-\alpha_j)}\le C_{10}\,x^{-(\alpha+2\beta)}\le C_{11},$$

or

$$\left|K_3(x,t)\right|\le C_{11}. \tag{62}$$

By virtue of (6), (10) and taking into account $x\le t\le1$(52), it follows that

$$\left|K_4(x,t)\right|\le\frac{(1-2\beta)\gamma_3}{k_2}\left(\frac{x}{t}\right)^{1-2\beta}\frac{C_{12}}{C_5(t-x)^{2\beta}}\le C_{13}(t-x)^{-2\beta},$$

or

$$\left|K_4(x,t)\right| \le C_{13}(t-x)^{-2\beta} \le C_{14}. \tag{63}$$

Now let's estimate the right-hand side of (47). Differentiating (33) with respect to y and setting, $y=0$ we obtain

$$\frac{\partial q_2(\xi,\eta;t,0)}{\partial y} = k_2\eta\left[(\xi-t)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\beta-1}. \tag{64}$$

Substituting (64) into (53) we have

$$F(x) = \frac{(2\beta-1)\gamma_3 x^{1-2\beta}}{k_2}\left[\frac{b(x)}{a_0(x)} - k_2\int_0^l \frac{\eta(s)\chi(s)}{\left[(\xi(s)-t)^2 + \frac{4}{(m+2)^2}\eta^{m+2}(s)\right]^{1-\beta}}ds\right]. \tag{65}$$

From this it is clear that the function $F(x)-$ has derivatives of any order in the interval (0,1). Let us clarify the behavior of the function $F(x)$ and its derivative for $x\to 0$ and $x\to 1$.

Let's consider the expression

$$F_1(x) = k_2\int_0^l \frac{\eta(s)\chi(s)}{\left[(\xi(s)-t)^2 + \frac{4}{(m+2)^2}\eta^{m+2}(s)\right]^{1-\beta}}ds. \tag{66}$$

Let us estimate expression (66). By virtue of (9), for sufficiently small $x>0$ we have

$$\left|F_1(x)\right| \le k_2\int_{l-\varepsilon}^{l}|\chi(s)|\frac{\eta}{\left[(\xi-x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{1-\beta}}ds + O(1) <$$

$$< C_{15}\int_{l-\varepsilon}^{l}\frac{\eta}{\left[(\xi-x)^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{1-\beta}}ds + O(1).$$

From here, by virtue of (13), for sufficiently small $\varepsilon>0$ we obtain

$$\left|F_1(x)\right| < C_{16}\int_{l-\varepsilon}^{l}\frac{\eta^{\frac{m}{2}}\left|\frac{d\eta}{ds}\right|}{\left[x^2 + \frac{4}{(m+2)^2}\eta^{m+2}\right]^{\frac{1}{2}+\beta}}ds + O(1) < C_{17}\int_0^{\delta}\frac{d\eta}{\left[x^2+\eta^2\right]^{\frac{1}{2}+\beta}} + O(1). \tag{67}$$

Having made the substitution $\mu^2=\omega$ in (67) taking into account (44), (58) and formula [10, (2.17), (2.14), (2.22), pp. 10-13] we have

$$|F_1(x)| < \frac{\delta^2}{x^{2\beta+1}}\left|\frac{\Gamma(1,5)\Gamma(\beta)}{\Gamma(0,5+\beta)}\right|\left(\frac{\delta^2}{x^2}\right)^{-\frac{1}{2}} +$$

$$+\frac{\delta}{x^{2\beta+1}}\left|\frac{\Gamma(1,5)\Gamma(-\beta)}{\Gamma(0,5)\Gamma(1-\beta)}\right|(x^2+\delta^2)^{-\beta}x^{2\beta+1}F\left(\beta,\ \frac{1}{2},1+\beta;\frac{x^2}{x^2+\delta^2}\right) <$$

$$<\frac{\delta}{x^{2\beta}}\left|\frac{\Gamma(1,5)\Gamma(\beta)}{\Gamma(0,5+\beta)}\right|+\left|\frac{\delta\,\Gamma(1,5)\Gamma(-\beta)}{\Gamma(0,5)\Gamma(1-\beta)}\right|(x^2+\delta^2)^{-\beta} < C_{18}\,x^{-2\beta},$$

or

$$|F_1(x)| < C_{18}x^{-2\beta}. \tag{68}$$

If $1-x$ it is small enough, then we find similarly

$$|F_1(x)| = C_{19}(1-x)^{-2\beta} \tag{69}$$

Carrying out the same reasoning, we obtain

$$|F_1'(x)| < C_{20}x^{-2\beta-1},\ |F_1'(x)| = C_{21}(1-x)^{-2\beta-1}. \tag{70}$$

By virtue of (6), (10), (68), (69), (70) from (65) we conclude that

$$F(x) \in C(\bar{J})\ \mathrm{I}\ \ C^1(J), \tag{71}$$

and $F'(x)$ the function becomes infinite in order of less than $2\beta+1$ when $x \to 1$, and is bounded when $x \to 0$.

By changing variables

$$\zeta = \frac{t^2}{1-2t+2t^2}, \qquad z = \frac{x^2}{1-2x+2x^2}, \tag{72}$$

let us bring equation (47) to the form

$$\omega(z) + \gamma_3\int_0^1 \frac{\omega(\zeta)d\zeta}{\zeta - z} - \int_0^1 \bar{K}(z,\zeta)\omega(\zeta)d\zeta = \text{℘}(z), \tag{73}$$

where

$$\omega(z) = (1-2x+2x^2)\text{℘}(x), \qquad \text{℘}(z) = (1-2x+2x^2)F(x),$$

$$\bar{K}(z,\zeta) = \frac{1-2t+2t^2}{2t(1-t)(1-2x+2x^2)}K(x,t) + \gamma_3\frac{(1-2x+2x^2)(1-2t+2t^2)}{(1-t)(t+x-2xt)},$$

$$x = \frac{\sqrt{z}}{\sqrt{z}+\sqrt{1-z}}, \qquad t = \frac{\sqrt{\zeta}}{\sqrt{\zeta}+\sqrt{1-\zeta}}.$$

Since $1+\gamma_3^2 \neq 0$ equation (47) is of normal type, its index is zero in the class h_2 [11] of functions $\omega(z) \in H(0,1)$, bounded at the ends of the interval $\bar{J}$.

We apply the Carleman-Vekua regularization method [11] developed by S.G. Mikhlin [12] and M.M. Smirnov [10, p. 258] to equation (73) and obtain a Fredholm integral equation of the second kind, the solvability of which follows from the uniqueness of the solution to the problem CK. Consequently, from the notation, $\text{℘}(x) = x^{1-2\beta}T(x)$, $\omega(z) = (1-2x+2x^2)\text{℘}(x)$ we find the function $T(x)$, where the function $T(x)$ is continuous in (0,1) and integrable in $[0,1]$.

Substituting the solution of the Fredholm $T(x)$ integral equation of the second kind into (28), we find $\tau(x)$. Next, knowing the function $\tau(x)$, we reconstruct the solution of the problem CK for equation (1) in the domain D as the solution of the problem DK for equation (1) with conditions (3) and (27).

Thus, the existence of a solution to the problem CK when $\delta(s) \neq 0$ proven.

Theorem 2 is proven.

5. Conclusion

In this paper, we study a boundary value problem with a conormal derivative on the boundary curve and a nonlocal integral condition with fractional Riemann-Liouville integrals on the degeneracy line for a degenerate elliptic equation of the second kind with a degeneracy index from minus one to zero.

The main results of the work are as follows.

1. Using the energy integral method, a uniqueness theorem for the solution to the problem is proved under the condition that the product of the boundary condition coefficients with the conormal derivative is nonnegative and that the coefficients of the nonlocal condition on the degeneracy line are of definite sign (Theorem 1). It is shown that if the coefficient of the desired function in the boundary condition does not vanish, uniqueness can also be established using the maximum principle.
2. An auxiliary problem with a given solution trace on the degeneracy line is introduced and studied. Its Green's function is constructed explicitly through the fundamental solution of the equation and the Gaussian hypergeometric function, the uniqueness of the solution is proven, and its integral representation is obtained.
3. Lemmas on the transformation of fractional integrals with singularities of power type are proved, which make it possible to derive a functional relationship between the trace of a solution on the degeneration line and its normal derivative, transferred from the domain to the degeneration line.
4. It is shown that eliminating the normal derivative from the functional relations obtained from the solution of the auxiliary problem and from the nonlocal condition leads to a singular integral equation of normal type with zero index in the class of bounded functions. Using the Carleman-Vekua regularization method, this equation is reduced to a Fredholm equation of the second kind, the solvability of which follows from the uniqueness of the solution of the original problem.
5. Thus, the existence theorem for a solution to the problem with a non-zero coefficient at the conormal derivative is proven (Theorem 2), and the solution to the original problem is restored as a solution to the auxiliary problem with the trace found on the degeneration line.

The obtained results complement the theory of boundary value problems with a conormal derivative for degenerate elliptic equations and open the possibility of further study of similar problems for mixed-type equations with multiple degeneracy lines, as well as problems with fractional differentiation operators in the boundary conditions. Of particular interest are cases where the coefficient of the conormal derivative vanishes on part of the boundary, variable orders of fractional integration, and generalizations of the obtained results to the multidimensional case.

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