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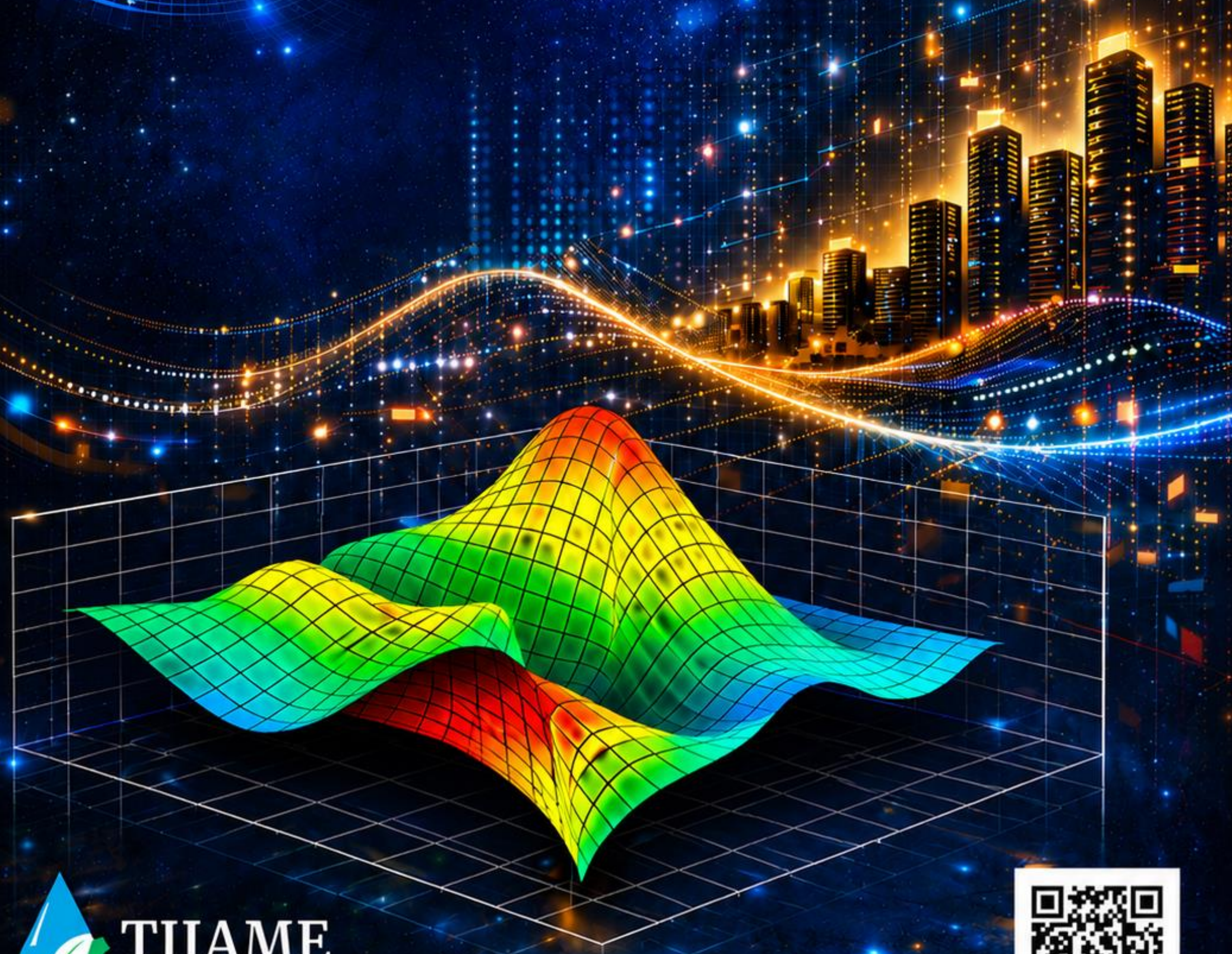

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Some identities involving hypergeometric functions

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ABSTRACT

This paper focuses on formulating and proving some identities involving hypergeometric functions. The main result generalizes an identity previously proven for specific cases, specifically for $m=1$ and $m=2$ with $b=a-1$. To prove the primary identity, the Laplace transform method is employed. The mathematical proof also utilizes the notations introduced by Srivastava and Daoust, as well as the properties of the Kampé de Fériet function. Additionally, a second identity is formulated and presented. These generalized identities can be actively applied to analyze and solve various boundary-value problems in the theory of differential equations.

1. Introduction

Hypergeometric functions have a wide range of practical applications in physics, engineering, mathematical statistics, and various other scientific fields. Notably, they are extensively used as a powerful mathematical tool for analyzing and solving complex problems within the theory of differential equations.

In the existing literature, numerous boundary-value problems and fundamental solutions have been successfully constructed by utilizing the unique properties of hypergeometric functions [1-6]. For instance, Salakhitdinov and Urinov explored eigenvalue problems for mixed-type equations featuring two singular coefficients [1]. Similarly, hypergeometric functions have been instrumental in finding fundamental solutions for the generalized bi-axially symmetric Helmholtz equation [3, 4] and certain classes of three-dimensional elliptic equations with singular coefficients [5].

Furthermore, obtaining explicit solutions to specific boundary-value problems often requires the use of mathematical identities containing hypergeometric functions [7-11]. Such identities have played a crucial role in solving the Cauchy problem for degenerating hyperbolic equations of the second kind [7, 9]. They have also been actively applied to the Tricomi problem for elliptic-hyperbolic and parabolic-hyperbolic type equations [8, 10] , as well as for solving non-local boundary-value problems [11].

The present work builds directly upon these foundational studies. Specifically, the core identity (1) discussed in this paper was initially proven by N. K. Mamadaliev for the specific case where

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$m=1$ and $b=a-1$ [8]. In a subsequent study, this result was extended and proven for the case where $m=2$ and $b=a-1$ [10]. Motivated by these previous achievements, this article aims to formulate and rigorously prove a generalized identity involving hypergeometric functions, which can further broaden their application in the theory of differential equations.

2. Preliminaries and Notation

To establish our main identities, we first need to recall some foundational definitions and notations. Multiple hypergeometric series can often become cumbersome to write and manipulate. However, they can be elegantly and compactly represented using the generalized Kampé de Fériet function. According to the notation formally introduced by Srivastava and Daoust (1969), this function in multiple variables is defined as follows:

$$F^{A:B^{(1)};...;B^{(n)}}_{C:D^{(1)};...;D^{(n)}}\left(\begin{matrix}\left[a:\theta^{(1)},...,\theta^{(n)}\right]:\left[b^{(1)}:\varphi^{(1)}\right];...;\left[b^{(n)}:\varphi^{(n)}\right];\\ x_1,...,x_n\\ \left[c:\psi^{(1)},...,\psi^{(n)}\right]:\left[d^{(1)}:\delta^{(1)}\right];...;\left[d^{(n)}:\delta^{(n)}\right];\end{matrix}\right)=\sum_{m_1,...,m_n=0}^{\infty}\Omega\left(m_1,...,m_n\right)\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

where the multiple sequence $\Omega\left(m_1,...,m_n\right)$ is given by the quotient of products of the Pochhammer symbols:

$$\Omega\left(m_1,...,m_n\right)=\frac{\prod_{j=1}^{A}\left(a_j\right)_{m_1\theta_j^{(1)}+...+m_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}\left(b_j^{(1)}\right)_{m_1\varphi_j^{(1)}}...\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{m_n\varphi_j^{(n)}}}{\prod_{j=1}^{C}\left(c_j\right)_{m_1\psi_j^{(1)}+...+m_n\psi_j^{(n)}}\prod_{j=1}^{D^{(1)}}\left(d_j^{(1)}\right)_{m_1\delta_j^{(1)}}...\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}\right)_{m_n\delta_j^{(n)}}}$$

By utilizing this compact notation, we can transform complex multiple sums into a more manageable and recognizable form. Specifically, if we expand the left-hand side of our proposed identity (1) into a power series and systematically rearrange the terms, it can be sequentially rewritten in the following manner:

$$\sum_{k_1=0}^{n_1}...\sum_{k_m=0}^{n_m}\prod_{i=1}^{m}\binom{n_i}{k_i}\frac{(-1)^k(1/2-b)_k}{(n+a)_k}(4x)^kF(n-k+b,-n+k-b+1,n+k+a;x)=$$
$$=\sum_{k_1=0}^{n_1}...\sum_{k_m=0}^{n_m}\sum_{l=0}^{\infty}\prod_{i=1}^{m}\frac{(-n_i)_{k_i}(1/2-b)_k}{(n+a)_k}\frac{(n-k+b)_l(-n+k-b+1)_l}{(n+k+a)_l}\frac{(4x)^{k_i}}{k_i!}\frac{x^l}{l!}=$$
$$=\sum_{k_1=0}^{n_1}...\sum_{k_m=0}^{n_m}\sum_{l=0}^{\infty}\prod_{i=1}^{m}\frac{(-n_i)_{k_i}(1/2-b)_k}{(n+a)_{k+l}}\frac{(-1)^l(1-n-b)_{k+l}}{(1-n-b)_{k-l}}\frac{(4x)^{k_i}}{k_i!}\frac{x^l}{l!}=$$
$$=F^{2:m;...;0}_{2:0;...;0}\left(\begin{matrix}[1/2-b:1,...,1,0],[1-n-b:1,...,1,1]:[-n_1:1],...,[-n_m:1];-;...;-\\ 4x,...,4x,-x\\ [n+a:1,...,1,1],[1-n-b:1,...,1,-1]:---------;-;...;-;\end{matrix}\right).$$

This preliminary transformation is crucial, as it sets the stage for the rigorous proof of our main theorems using integral transforms.

3. Main Results

In this section, we present the primary findings of our research. The core result is a generalized mathematical identity that equates a complex multiple summation involving Gaussian hypergeometric functions to a single hypergeometric function.

Theorem 1. Let the real parameters satisfy $a\geq b>0$ and let $x\geq 0$. Then, the following identity holds true:

$$\sum_{k_1=0}^{n_1}...\sum_{k_m=0}^{n_m}\prod_{i=1}^{m}\binom{n_i}{k_i}\frac{(-1)^k(1/2-b)_k}{(n+a)_k}(4x)^kF(n-k+b,-n+k-b+1,n+k+a;x)=$$
$$=F(1+n-b,-n+b,n+a;x). \quad (1)$$

In the formulation above, the integer parameters are defined such that $m, n_j \in \mathbb{N}$ for $j = 1, 2, ..., m$. Furthermore, the sums of the respective indices are denoted concisely as $n = n_1 + ... + n_m$ and $k = k_1 + ... + k_m$. The binomial coefficients and the Pochhammer symbol (rising factorial) utilized in the identity are defined in the standard mathematical manner:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = (-1)^k \frac{(-n)_k}{k!}$$

$$(a)_k = \begin{cases} 1, & k = 0; \\ a(a+1)\cdot ... \cdot (a+k-1), & k \in \mathbb{N}. \end{cases}$$

Following a similar logical framework and methodology, we can establish a second significant identity, which further broadens the scope of our results.

Theorem 2. Assuming that the variable $z \geq 0$ and the parameters satisfy the condition $\pm a, 2b \neq -1, -2, ...$, the following equality is satisfied:

$$\sum_{k_1=0}^{n_1} ... \sum_{k_m=0}^{n_m} \prod_{i=1}^{m} \binom{n_i}{k_i} \frac{(a-n)_k}{(1/2+b)_k} \left(\frac{z}{2}\right)^{2k} F\left(c+k, a+b+2k-2n; 2b+2k; z\right) =$$

$$= \frac{\Gamma(2b)\Gamma(a)}{\Gamma(a+b)\Gamma(b)} z^{1-2b} (1-z)^{-a+n} F(1-a-b, 1-b; 1-a; 1-z).$$

4. Proofs of the Main Results

In this section, we provide the detailed proof for our primary result.

Proof of Theorem 1. To prove Theorem 1, we employ the Laplace transform method. Let us rewrite the left-hand side of identity (1) in the following form:

$$= x^{-a-n+1} B(a, b, n; x) \tag{2}$$

where the function $B(a, b, n; x)$ is defined as:

$$B(a, b, n; x) = \sum_{k_1=0}^{n_1} ... \sum_{k_m=0}^{n_m} \prod_{i=1}^{m} \binom{n_i}{k_i} \frac{(-1)^k (1/2-b)_k}{(n+a)_k} 4^k \times$$

$$\times x^{a+n+k-1} F(n-k+b, -n+k-b+1, n+k+a; x) \tag{3}$$

By making the substitution $x = 1 - e^{-t}$ (where $t > 0$) in equation (3), we obtain:

$$B(a, b, n; 1-e^{-t}) = \sum_{k_1=0}^{n_1} ... \sum_{k_m=0}^{n_m} \prod_{i=1}^{m} \binom{n_i}{k_i} \frac{(-1)^k (1/2-b)_k}{(n+a)_k} 4^k \times$$

$$\times (1-e^{-t})^{a+n+k-1} F(n-k+b, -n+k-b+1, n+k+a; 1-e^{-t}).$$

Next, let us introduce the Laplace transform of this function with respect to the variable t:

$$\Phi(a, b; n; p) = \int_0^{+\infty} e^{-pt} B(a, b, n; 1-e^{-t}) dt \leftrightarrow B(a, b, n; 1-e^{-t}).$$

Using the standard tables of integral transforms (e.g., Bateman and Erdélyi [13]), we use the following known operational relation:

$$(1-e^{-t})^{c-1} F(a, b, c; 1-e^{-t}) \leftrightarrow \frac{\Gamma(p)\Gamma(c-a-b+p)\Gamma(c)}{\Gamma(c-a+p)\Gamma(c-b+p)} \tag{4}$$

Applying formula (4), we find the image $\Phi(a, b; n; p)$ as:

$$\Phi(a, b; n; p) = \sum_{k_1=0}^{n_1} ... \sum_{k_m=0}^{n_m} \prod_{i=1}^{m} \binom{n_i}{k_i} \frac{(-1)^k (1/2-b)_k}{(n+a)_k} 4^k \times$$

$$\times \frac{\Gamma(p)\Gamma(a+k+n+p-1)\Gamma(a+n+k)}{\Gamma(a+2k-b+p)\Gamma(a+2n+b+p-1)}.$$

To simplify the gamma terms, we utilize the generalized reduction formula:

$$\Gamma(n+\gamma) = (\gamma)_n \Gamma(\gamma), \quad \gamma > 0 \tag{5}$$

along with the identity:

$$(a)_{2k}=4^k(a/2)_k(a/2+1/2)_k. \tag{6}$$

Using these relations, we can rewrite $\Phi(a,b;n;p)$ in terms of the Kampé de Fériet function:

$$\Phi(a,b;n;p)=\frac{\Gamma(p)\Gamma(a+n+p-1)\Gamma(a+n)}{\Gamma(a-b+p)\Gamma(a+2n+b+p-1)}\times$$

$$\times F^{2:1;\ldots;1}_{2:0;\ldots;0}\left(\begin{matrix}[1/2-b,a+n+p-1]:[-n_1];\ldots;[-n_m]\\ 1,\ldots,1\\ [a/2-b/2+p/2,a/2-b/2+p/2+1/2]:-;\ldots;-\end{matrix}\right).$$

By applying the reduction formula for the Kampé de Fériet function (as provided in [12, p. 39, eq. 32]), this multiple series collapses into a generalized hypergeometric series ${}_3F_2$:

$$\Phi(a,b;n;p)=\frac{\Gamma(p)\Gamma(a+n+p-1)\Gamma(a+n)}{\Gamma(a-b+p)\Gamma(a+2n+b+p-1)}\times$$

$$\times {}_3F_2\left(\begin{matrix}-n,1/2-b,a+n+p-1;\\ a/2-b/2+p/2,a/2-b/2+p/2+1/2\end{matrix}\,1\right)$$

Next, we evaluate the ${}_3F_2$ series at unit argument using Prudnikov's formula (see [14, p. 454, eq. 87]):

$$\Phi(a,b;n;p)=\frac{\Gamma(p)\Gamma(a+n+p-1)\Gamma(a+n)}{\Gamma(a-b+p)\Gamma(a+2n+b+p-1)}\times$$

$$\times\frac{(a/2+b/2+p/2-1/2)_n(-a/2-b/2-p/2-n+1)_n}{(a/2-b/2+p/2)_n(-a/2+b/2-p/2+1/2-n)_n}$$

Applying the property $(a-n)_n=(-1)^n(1-a)_n$, together with formulas (5) and (6), the expression drastically simplifies to:

$$\Phi(a,b;n;p)=\frac{\Gamma(p)\Gamma(a+n+p-1)\Gamma(a+n)}{\Gamma(a-b+p+2n)\Gamma(a+b+p-1)}.$$

Now, we find the original function (inverse Laplace transform) according to our previously stated relation (4):

$$\Phi(a,b;n;p)\leftrightarrow(1-e^{-t})^{a+n-1}F(b-n,1-b+n;a+n;1-e^{-t})=B(a,b,n;1-e^{-t}).$$

Making the reverse substitution $e^{-t}=1-x$, we obtain the explicit form for B(a,b,n;x):

$$B(a,b,n;x)=x^{a+n-1}F(b-n,1-b+n;a+n;x).$$

Finally, substituting this resulting expression for $B(a,b,n;x)$ back into equation (2) directly yields the right-hand side of identity (1). This concludes the proof of Theorem 1.

The proof of Theorem 2 proceeds along the same logical lines and methodology as the proof of Theorem 1. By rewriting the multiple summation and applying the Laplace transform method, the expression can be similarly represented in terms of the Kampé de Fériet function. Utilizing the relevant operational relations for integral transforms and the generalized reduction formulas for hypergeometric series, the complex series simplifies directly to the explicit form presented on the right-hand side. Since the underlying analytical steps are strictly analogous to those demonstrated in detail for Theorem 1, they are omitted here for the sake of brevity.

5. Conclusion

In this paper, we have successfully formulated and rigorously proven generalized mathematical identities involving Gaussian hypergeometric functions. By systematically employing the Laplace transform method and utilizing the properties of the generalized Kampé de Fériet function, we demonstrated that complex multiple summations can be elegantly reduced to a single hypergeometric function.

These main results significantly extend previous foundational studies in this area, specifically generalizing the identities that were previously established only for restricted cases (such as $m=1$ and $m=2$). The newly proven identities not only contribute to the theoretical development of special functions but also hold substantial potential for broader mathematical applications. In particular, we anticipate that these generalized identities will be actively utilized as an effective analytical tool for constructing fundamental solutions and solving various complex boundary-value problems within the theory of differential equations.

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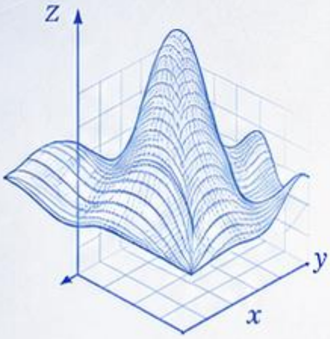


$$\frac{dx}{dt} = Ax + b(t)$$

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u + f$$

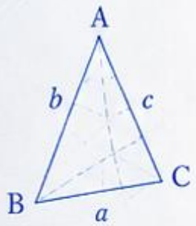
$$\sum_{i=1}^{n} a_i x_i + b$$

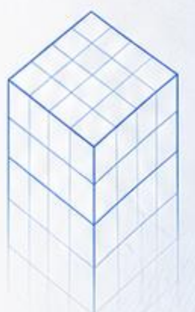
$$f(x) = \int_a^b f(x)\, dx$$

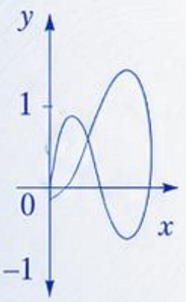
CONTENTS

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\sum_{i=1}^{n} w_i x_i$$







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