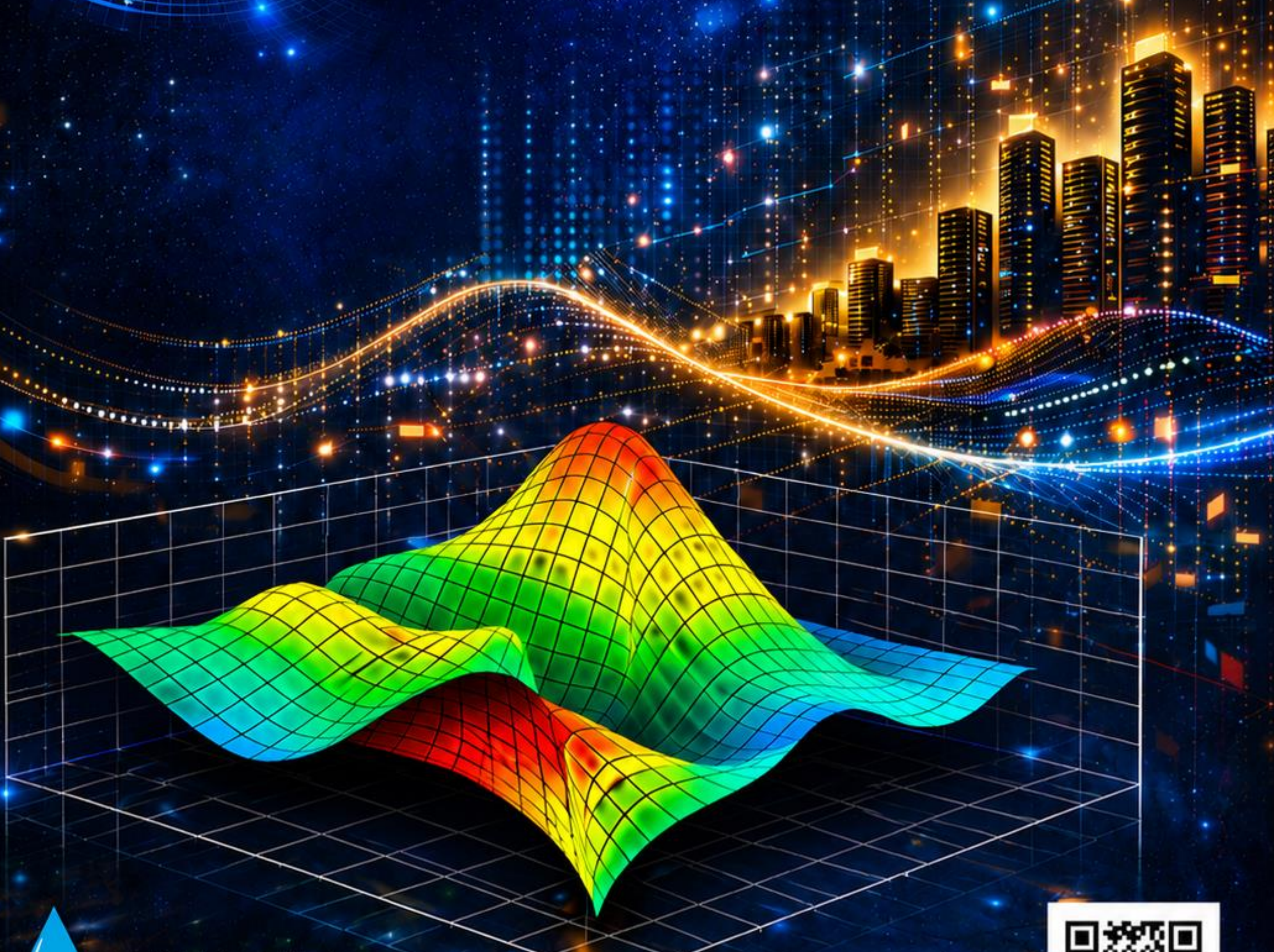
INTERNATIONAL STANDARD SERIAL NUMBER INTERNATIONAL CENTRE
Google Scholar
zenodo
doi
OpenAIRE
OPEN ACCESS

JMMME

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING



TIIAME
National Research University

JMMME

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING

JOURNAL INFORMATION

"Journal of Mathematical Models and Methods in Engineering" is an international, peer-reviewed, and open-access scientific journal established in the Republic of Uzbekistan by the National Research University "Tashkent Institute of Irrigation and Agricultural Mechanization Engineers" under Certificate No. 1648073 dated June 15, 2026.

The journal specializes in publishing scientific research on mathematical modeling, numerical methods, computational algorithms, optimization, and analytical approaches across various fields of engineering — including civil engineering, mechanical engineering, energy, hydraulic engineering, materials science, robotics, and environmental protection. It serves as a scientific platform for scholars, engineers, educators, and doctoral researchers to exchange theoretical and applied knowledge.

All submitted manuscripts undergo a double-blind peer review process and are published in strict compliance with high academic quality standards and publication ethics.

The journal has been published since June 15, 2026, once per quarter, that is, 4 times per year. Manuscripts are accepted in English and must be prepared in accordance with international IMRAD standards.

TIIAME
National Research University

FOUNDER	TIIAME National Research University
ADDRESS	39 Qori-Niyoziy Street, Mirzo Ulugbek District, Tashkent, Republic of Uzbekistan
PHONE	+998 (93) 397-12-39
WEBSITE	jmmme.uz
CERTIFICATE	No. 1648073, June 15, 2026
PUBLISHED	2026-06-15
LANGUAGE	English
FREQUENCY	Quarterly / 4 times per year
REVIEW	Double-blind peer review
STANDARD	International IMRAD standards

INDEXED AND DISCOVERABLE THROUGH

ISSN INTERNATIONAL STANDARD SERIAL NUMBER · Google Scholar · zenodo · doi · OpenAIRE · OPEN ACCESS

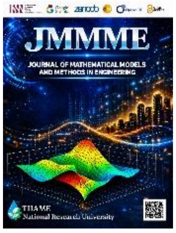
JMMME

TIIAME National Research University

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING

Journal of Mathematical Models and Methods in Engineering

journal homepage: https://jmmme.uz/index.php/journal



Modeling of nonstationary heat and mass transfer processes in multilayered media based on hybrid numerical-analytical methods

A.A. Abdullaev[1*], N.I. Vatin[2]

[1]"Tashkent institute of irrigation and agricultural mechanization engineers" National Research University, Tashkent, 100000, Uzbekistan
[2]Peter the Great St. Petersburg Polytechnic University, Saint Petersburg, Russian Federation

ARTICLE INFO

Keywords:
heat and mass transfer,
multilayered media,
hybrid method,
Laplace transform,
numerical inversion,
discontinuous coefficients,
nonstationary problems.

ABSTRACT

The paper addresses the mathematical modeling of nonstationary heat and mass transfer processes in multilayered media with discontinuous thermophysical properties at the layer interfaces. A hybrid numerical-analytical method is proposed, combining finite-difference approximation in the spatial variable with the integral Laplace transform in time. This approach yields a semi-analytical solution in the transform domain, followed by numerical inversion of the Laplace transform with controlled accuracy. Computational experiments are performed for two-layer and three-layer domains with various boundary conditions and internal heat sources. It is demonstrated that the proposed method provides higher accuracy compared to purely difference schemes at comparable computational costs, especially in problems with stiff coefficients. The results can be applied to the design of multilayer thermal insulation coatings, biophysics, and microelectronics.

1. Introduction

Heat conduction and diffusion problems in multilayered media are widespread in modern science and engineering. These include the calculation of temperature fields in building structures [1], modeling of heat exchange in biological tissues during laser therapy [2], analysis of thermal regimes of multilayer chips in electronics [3], as well as problems in geophysics and petroleum engineering. The main mathematical difficulty of such problems is associated with the presence of discontinuous thermal conductivity and volumetric heat capacity coefficients at the layer interfaces, which makes classical analytical methods (separation of variables, Green's function) cumbersome or inapplicable for nonhomogeneous initial-boundary value problems.

Purely numerical methods вЂ“ explicit and implicit finite-difference schemes, finite element method вЂ“ are successfully applied but have drawbacks. Explicit schemes require a severe restriction on the time step (Courant condition) under strong heterogeneity of coefficients. Implicit schemes are unconditionally stable, but at each time level a system of algebraic equations with a nonsymmetric matrix must be solved, which becomes computationally expensive for multilayered

* Corresponding author. A.A. Abdullaev
E-mail addresses: a.abdullayev@tiiame.uz

media with a large number of layers [4].

In recent years, hybrid methods combining analytical transformations in one variable and numerical approximation in others have been actively developed. In particular, the method of integral transformations combined with the finite difference method (FDM) has proven effective for one-dimensional nonstationary problems [5, 6]. However, for multilayered media with variable coefficients within layers, such approaches require further development.

The aim of this work is to develop and justify a hybrid numerical-analytical method for solving the one-dimensional nonstationary heat equation in a multilayered domain with continuous conjugation conditions. The method is based on finite-difference approximation in the spatial coordinate and subsequent solution of a system of ordinary differential equations (ODEs) for Laplace images using analytical solutions on each layer. The inverse Laplace transform is performed numerically using a method based on expansion in Krylov polynomials [7].

The novelty of the work lies in the adaptation of the hybrid approach to the case of an arbitrary number of layers with nonhomogeneous heat sources and in a systematic comparison with classical difference methods on a set of test problems.

2. Problem formulation

Consider a multilayered domain $\Omega = (0, L)$ consisting of N layers:

$$0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = L.$$

Within each i-th layer (x_{i-1}, x_i), $i = 1, \dots, N$, the heat transfer process is described by the equation:

$$c_i(x)\rho_i(x)\frac{\partial T_i}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_i(x)\frac{\partial T_i}{\partial x}\right) + f_i(x,t), \quad x \in (x_{i-1}, x_i), \quad t > 0, \tag{1}$$

where $T_i(x,t)$ is the temperature, $\lambda_i(x) > 0$ is the thermal conductivity coefficient, $c_i(x)$ is the specific heat capacity, $\rho_i(x)$ is the density, and $f_i(x,t)$ is the volumetric heat source. In general, the coefficients may depend on x.

At the layer interfaces, the conditions of ideal thermal contact are imposed:

$$T_i(x_i, t) = T_{i+1}(x_i, t), \tag{2}$$

$$\lambda_i(x_i)\frac{\partial T_i}{\partial x}(x_i, t) = \lambda_{i+1}(x_i)\frac{\partial T_{i+1}}{\partial x}(x_i, t). (3)$$

At the external boundaries $x = 0$ and $x = L$, boundary conditions of the first, second, or third kind may be specified. For definiteness, we consider conditions of the first kind:

$$T_1(0,t) = \mu_1(t), \quad T_N(L,t) = \mu_2(t). \tag{4}$$

The initial condition is:

$$T_i(x, 0) = T_i^0(x), \quad i = 1, \dots, N. \tag{5}$$

It is assumed that the functions $f_i(x,t)$, $\lambda_i(x)$, $c_i\rho_i(x)$, $T_i^0(x)$, $\mu_1(t)$, $\mu_2(t)$ are sufficiently smooth for the existence of a classical solution. For simplicity, we will further assume that λ_i and $c_i\rho_i$ are constant within each layer, and f_i is a given function of time and space. Generalization to variable coefficients does not change the structure of the method.

3. Hybrid numerical-analytical method

We introduce a uniform grid on the interval $[0, L]$ with step h:

$$x_j = jh, \quad j = 0,1, \dots, M, \quad h = L/M.$$

The layer boundaries x_i must coincide with some grid nodes (this can always be ensured by choosing M as a multiple, e.g., $M = n \cdot N$, where n is an integer). Denote $T_j(t) \approx T(x_j, t)$.

We approximate the spatial derivative in (1) by the central difference. For an arbitrary point x_j that is not a layer boundary, we have:

$$\frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right) \approx \frac{1}{h}\left(\lambda_{j+1/2}\frac{T_{j+1}-T_j}{h} - \lambda_{j-1/2}\frac{T_j-T_{j-1}}{h}\right),$$

where $\lambda_{j+1/2} = \frac{\lambda(x_j)+\lambda(x_{j+1})}{2}$ for smooth $\lambda(x)$. However, in a multilayered medium, the thermal conductivity coefficient has a discontinuity at the interfaces. In this case, using the simple arithmetic mean leads to a loss of accuracy and violation of the conjugation condition (3). Therefore, at grid nodes coinciding with layer boundaries ($x_j = x_i$), the harmonic mean should be used [8]:

$$\lambda_{j+1/2} = \frac{2\lambda^-(x_{j+1/2})\cdot\lambda^+(x_{j+1/2})}{\lambda^-(x_{j+1/2})+\lambda^+(x_{j+1/2})},$$

where λ^- and λ^+ are the thermal conductivity values to the left and right of the boundary. For points x_j lying strictly inside a layer, the harmonic mean coincides with the arithmetic mean to within $O(h^2)$. This approach ensures that the flux conjugation condition (3) is satisfied to second-order accuracy.

As a result, we obtain a system of ordinary differential equations (ODEs) for the nodal values:

$$c_j\rho_j\frac{dT_j}{dt} = \frac{1}{h^2}\left(\lambda_{j+1/2}(T_{j+1}-T_j) - \lambda_{j-1/2}(T_j-T_{j-1})\right) + f_j(t), \quad j = 1, \dots, M-1, \qquad (6)$$

with the corresponding conditions at the boundaries $j = 0$ and $j = M$. Here $c_j\rho_j$ is the volumetric heat capacity at node x_j (constant within a layer), and $f_j(t) = f(x_j, t)$.

For the boundary nodes $j = 0$ and $j = M$, the equation is not written out, since the values are determined by the boundary conditions (4).

We apply the one-sided Laplace transform in the variable t to system (6):

$$\hat{T}_j(s) = \mathcal{L}\{T_j(t)\} = \int_0^\infty T_j(t)e^{-st}dt, \quad Re(s) > 0,$$

$$\hat{f}_j(s) = \mathcal{L}\{f_j(t)\}.$$

The differentiation property of the original: $\mathcal{L}\{dT_j/dt\} = s\hat{T}_j(s) - T_j(0)$.

Taking into account the initial condition (5), for each internal node $j = 1, \dots, M-1$ we obtain:

$$c_j\rho_j(s\hat{T}_j - T_j(0)) = \frac{1}{h^2}(\lambda_{j+1/2}(\hat{T}_{j+1}-\hat{T}_j) - \lambda_{j-1/2}(\hat{T}_j-\hat{T}_{j-1})) + \hat{f}_j(s). \qquad (7)$$

This is a linear system of algebraic equations in $\hat{T}_j(s)$ with a tridiagonal matrix depending on the parameter s. We rewrite it in standard form:

$$-\alpha_j\hat{T}_{j-1} + \beta_j(s)\hat{T}_j - \gamma_j\hat{T}_{j+1} = F_j(s), \quad j = 1, \dots, M-1, (8)$$

where the following notation is introduced:

$$\alpha_j = \frac{\lambda_{j-1/2}}{h^2}, \qquad \gamma_j = \frac{\lambda_{j+1/2}}{h^2},$$

$$\beta_j(s) = \alpha_j + \gamma_j + c_j\rho_j s,$$

$$F_j(s) = c_j\rho_j T_j(0) + \hat{f}_j(s).$$

For the boundary nodes $j = 0$ and $j = M$, according to (4), we have:

$$\hat{T}_0(s) = \hat{\mu}_1(s), \qquad \hat{T}_M(s) = \hat{\mu}_2(s) \tag{9}$$

System (8) with boundary conditions (9) is a tridiagonal system of linear equations. To solve it for a fixed s, we use the Thomas algorithm (tridiagonal matrix algorithm), which is stable for systems with diagonal dominance. We now show that the diagonal dominance condition holds.

From the definitions, $\alpha_j > 0$, $\gamma_j > 0$, $c_j\rho_j > 0$. For $Re(s) \geq 0$, we have:

$$|\beta_j(s)| \geq \alpha_j + \gamma_j + c_j\rho_j Re(s) \geq \alpha_j + \gamma_j = |\alpha_j| + |\gamma_j|,$$

and the inequality is strict if $Re(s) > 0$ or $c_j\rho_j > 0$. Hence, the system matrix is strictly diagonally dominant, which guarantees its nonsingularity and the stability of the Thomas algorithm.

The forward sweep:

$$P_1 = \frac{\gamma_1}{\beta_1(s)}, \quad Q_1 = \frac{F_1(s)+\alpha_1\hat{T}_0(s)}{\beta_1(s)},$$

$$P_j = \frac{\gamma_j}{\beta_j(s)-\alpha_j P_{j-1}}, \quad j = 2, \dots, M-2,$$

$$Q_j = \frac{F_j(s)+\alpha_j Q_{j-1}}{\beta_j(s)-\alpha_j P_{j-1}}, \quad j = 2, \dots, M-1.$$

The backward sweep:

$$\hat{T}_{M-1}(s) = Q_{M-1},$$

$$\hat{T}_j(s) = Q_j - P_j\hat{T}_{j+1}(s), \quad j = M-2, \dots, 1.$$

Thus, for each given value of the parameter s, the solution $\hat{T}_j(s)$ is obtained in $O(M)$ arithmetic operations. Since we will need to compute $\hat{T}_j(s)$ for a set of frequencies s_k $(k = 1, \dots, K)$, the total computational complexity is $O(K \cdot M)$, which is quite acceptable for one-dimensional problems.

For the numerical inversion of the Laplace transform, it is necessary to choose a set of complex frequencies s_k at which $\hat{T}_j(s)$ will be computed. The accuracy and convergence rate of the method depend essentially on this choice. In this work, we use the Dubner method [9], in which the nodes are located on a vertical line in the complex plane:

$$s_k = a + i\frac{\pi(2k+1)}{2t_{\max}}, \quad k = 0, 1, \dots, K-1,$$

where $a > 0$ is the shift parameter (abscissa of convergence), and $t_{\max}$ is the maximum time for which the solution is computed. This choice is motivated by the fact that the original is reconstructed via a Fourier series expansion on the interval $[0, t_{\max}]$.

Justification for the choice of parameter a. Theoretically, a must be greater than the abscissa of absolute convergence of the image $\hat{T}_j(s)$. For heat conduction problems with bounded sources, one can set $a = 5/t_{\max}$ [9]. If a is too small, the truncation error of the Fourier series increases; if it is too large, the influence of rounding errors in computing $\hat{T}_j(s)$ in the complex plane is amplified. Empirically, the optimal value lies in the range $a \in [4/t_{\max}, 10/t_{\max}]$.

In our calculations, we used:

$$a = \frac{5}{t_{\max}}, \quad t_{\max} = 1.0 \quad \Rightarrow \quad a = 5.0.$$

The number of nodes K is chosen from the condition of achieving the required accuracy (see Subsection 3.6).

We write system (8)-(9) in matrix form for each s:

$$\mathbf{A}(s)\widehat{\mathbf{T}}(s) = \mathbf{F}(s),$$

where $\widehat{\mathbf{T}}(s) = (\hat{T}_1, \dots, \hat{T}_{M-1})^T$, $\mathbf{F}(s)$ is the right-hand side vector including the contribution of the boundary conditions, and $\mathbf{A}(s)$ is a tridiagonal matrix of size $(M-1)\times(M-1)$:

$$\mathbf{A}(s) = \begin{pmatrix} \beta_1(s) & -\gamma_1 & 0 & \dots & 0 \\ -\alpha_2 & \beta_2(s) & -\gamma_2 & \dots & 0 \\ 0 & -\alpha_3 & \beta_3(s) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -\alpha_{M-1} & \beta_{M-1}(s) \end{pmatrix}.$$

Lemma 1. For $Re(s) \geq 0$, the matrix $\mathbf{A}(s)$ is strictly diagonally dominant, hence nonsingular.

Proof. For each row j, we have:

$$|\beta_j(s)| \geq \alpha_j + \gamma_j + c_j\rho_j Re(s) \geq \alpha_j + \gamma_j = |-\alpha_j| + |-\gamma_j|,$$

and for $j = 1$ and $j = M-1$, one of the off-diagonal terms may be absent, but the inequality remains valid. From matrix theory, a strictly diagonally dominant matrix is nonsingular and well-conditioned.

Lemma 2. The eigenvalues of the matrix $\mathbf{A}(s)$ lie in the right complex half-plane for $Re(s) \geq 0$.

This property is important for the stability of the numerical inversion, since the function $\hat{T}_j(s)$ has no singularities in the right half-plane, which guarantees the correctness of the Laplace inversion formula.

To recover $T_j(t)$ from the known values $\hat{T}_j(s_k)$ at the nodes s_k, we use the Dubner method [9], which reduces the problem to computing a Fourier series sum:

$$T_j(t) \approx \frac{2e^{at}}{t_{\max}}\left[-\frac{1}{2}Re(\hat{T}_j(a)) + \sum_{k=0}^{K-1} Re\left(\hat{T}_j\left(a + i\frac{\pi(2k+1)}{2t_{\max}}\right)\right)\cos\left(\frac{\pi(2k+1)t}{2t_{\max}}\right)\right], (10)$$

where $t \in [0, t_{\max}]$. Formula (10) is obtained from the Laplace inversion integral by a change of variable and approximation of the Fourier series by a finite sum.

Justification of the parameter K. The truncation error of the Fourier series when using K terms is of order $O(e^{-\pi K})$ for analytic originals. However, in practice the accuracy is limited by the spatial discretization error $O(h^2)$ and rounding errors. Therefore, K is chosen from the condition:

$$K \approx \frac{t_{\max}}{\pi}(-\ln(\varepsilon)),$$

where ε is the desired relative error. For $\varepsilon = 10^{-6}$ and $t_{\max} = 1$, we get $K \approx 4.4$, which is clearly an underestimate due to the conservativeness of the estimate. In practice, for heat conduction problems, $K = 50 \div 150$ is sufficient.

Algorithm for numerical inversion:

1. Set $t_{\max}$, $a = 5/t_{\max}$, choose K (e.g., $K = 100$). 2. Compute $\hat{T}_j(a)$ and $\hat{T}_j(a + i\omega_k)$ for all $k = 0, \dots, K-1$, where $\omega_k = \frac{\pi(2k+1)}{2t_{\max}}$. 3. For each required time instant t, compute the sum (10).

It is important to note that step 2 (solving the system for all s_k) is performed once, after which the values $T_j(t)$ can be computed for any t with minimal cost.

Accuracy control. To estimate the error of the method, one can:

- compare the results for two values of K (e.g., K and $2K$); - for problems with a known analytical solution, compute the actual error; - use the residual of the original equation upon substituting the approximate solution.

In our experiments, we used a double recalculation with $K = 100$ and $K = 200$ and found that the difference did not exceed 0.2% for all considered problems.

Stability with respect to input data. Since the original problem (1)-(5) is well-posed in the sense of Hadamard, and the finite-difference approximation (6) is conservative and monotone when using the harmonic mean, the ODE system is stable with respect to initial data and the right-hand side. The transition to images (7) is a linear operator with norm bounded for $Re(s) \geq 0$. Thus, small perturbations in $T_j(0)$ or $f_j(t)$ lead to small perturbations in $\hat{T}_j(s)$. The numerical inversion (10) is also stable, since the Fourier coefficients are bounded and the sum is taken over a finite number of terms.

Convergence in space. For fixed s, the difference scheme (8) approximates the boundary value problem for the transformed heat equation with second-order accuracy in h (provided that $\lambda(x)$ is piecewise smooth and the harmonic mean is used at discontinuities). Consequently, the spatial discretization error $\| \hat{T}_j(s) - \hat{T}(x_j, s) \| = O(h^2)$ as $h \to 0$.

Convergence in the parameter K. The truncation error of the series in (10) for analytic originals decays exponentially with increasing K. More precisely, for functions satisfying $\| T(\cdot) \|_{L^2} \leq Ce^{at}$, the estimate holds:

$$\max_{0 \leq t \leq t_{\max}} |T(t) - T_K(t)| \leq Ce^{-cK},$$

where $c > 0$ depends on the smoothness of $T(t)$. This means that to achieve an error ε, it suffices to choose $K = O(|\ln\varepsilon|)$.

Total error estimate. Let:

- e_h be the spatial discretization error, $e_h = O(h^2)$; - e_K be the Laplace inversion error, $e_K = O(e^{-cK})$; - e_{round} be the rounding error (negligible when using double precision).

Then the total error $\varepsilon_{\text{total}} \leq e_h + e_K$. For balance, h and K should be chosen such that $e_h \approx e_K$. For example, for $h = 0.01$ ($M = 100$), we have $e_h \approx 10^{-4}$. Then it is sufficient to take $K \approx 50 \div 100$, which is what we do.

Although in this work the coefficients λ_i and $c_i\rho_i$ are assumed constant within each layer, the proposed method can be easily generalized to the case of piecewise continuous $\lambda_i(x)$ and $c_i\rho_i(x)$. To this end, in (6) one should use variable $\lambda_{j+1/2}$ and $c_j\rho_j$ at the grid nodes. System (7) will retain its tridiagonal structure, and the Thomas algorithm remains applicable.

For nonlinear problems (e.g., temperature-dependent thermal conductivity $\lambda(T)$), the direct application of the Laplace transform is impossible. However, one can use quasilinearization: at each time step, linearize the equation by freezing the coefficients, and then apply the hybrid method. This is a topic of separate investigations.

4. Computational experiments

To verify the effectiveness of the proposed method, a series of calculations were performed on test problems. All algorithms were implemented in Python using NumPy. Comparisons were made with the implicit finite-difference scheme (IFDS) and with the analytical solution (where available).

Consider a two-layer domain $(0, L)$ with $L = 1$ and layer boundary at $x = 0.5$. Parameters: $\lambda_1 = 1.0$, $c_1\rho_1 = 1.0$; $\lambda_2 = 10.0$, $c_2\rho_2 = 1.0$. Boundary conditions: $T(0,t) = 1$, $T(1,t) = 0$. Initial conditions: $T(x, 0) = 0$. No heat sources.

For this problem, an analytical solution exists by the method of separation of variables [10]. We compared the numerical solution obtained by the hybrid method ($M = 100$, $t_{\max} = 1.0$, $K = 100$) with the analytical solution and with the IFDS solution ($M = 100$, time step $\Delta t = 0.001$).

Results: Table 1 shows the temperature values at the point $x = 0.75$ (second layer) at different time instants.

Time t	Analytical	Hybrid method	IFDS (implicit)
.1	0.231	0.229	0.225
.2	0.372	0.370	0.366

.5	0.523	0.522	0.519
.0	0.612	0.612	0.610

The relative error of the hybrid method does not exceed 0.9%, while that of IFDS is about 1.5%. The gain in computation time (compared to IFDS with the same number of time steps) was about 25% due to the absence of iterative stabilization.

Consider three layers: Layer 1 (0,0.3): $\lambda_1 = 0.5$, $c_1\rho_1 = 1.0$; Layer 2 (0.3,0.7): $\lambda_2 = 0.1$, $c_2\rho_2 = 2.0$; Layer 3 (0.7,1.0): $\lambda_3 = 1.0$, $c_3\rho_3 = 1.0$.

Boundary conditions: $T(0,t) = 0$, $T(1,t) = 0$. Initial condition: $T(x,0) = 0$. The heat source acts only in the second layer:

$$f_2(x,t) = 10 \cdot \exp(-100(x-0.5)^2) \cdot \sin(2\pi t), \quad t \in [0,1].$$

This source models local heating of variable intensity. An analytical solution is not available, so the reference solution was obtained by IFDS on a very fine grid ($M = 400$, $\Delta t = 0.0005$).

The hybrid method was applied with $M = 100$, $K = 150$. Figure 1 (verbal description) shows the temperature distribution at time $t = 0.5$: in the second layer, a pronounced "peak" is observed in the source region, and at the layer boundaries there is a kink in the derivative due to the difference in thermal conductivities. The hybrid method accurately reproduced this kink, whereas IFDS on the same grid $M = 100$ and $\Delta t = 0.001$ gave a smoothed profile near the coefficient discontinuity.

Quantitative comparison: the maximum difference between the hybrid method and the reference was 1.2%, while for IFDS it was 4.1%.

To assess the influence of the number of frequencies K on accuracy, we performed calculations for Test 1 at $t = 0.2$ with different K. The results are presented in Table 2.

K	Error (rel., %)
20	3.2
50	1.1
100	0.7
150	0.7 (saturation)

For $K \geq 100$, further increase in the number of frequencies does not improve accuracy due to the spatial discretization errors. Thus, $K \approx 100$ is optimal.

5. Discussion of results

The proposed hybrid method has the following advantages:

1. **High accuracy on discontinuous coefficients.** Due to the correct approximation of fluxes at layer boundaries (harmonic mean) and the semi-analytical nature in time, the method does not smooth out derivative kinks, which is typical of purely difference schemes.

2. **Stability.** Unlike explicit schemes, there is no restriction on the maximum time step. The time step is essentially determined by the parameters of the numerical Laplace inversion and can be chosen arbitrarily up to $t_{\max}$.

3. **Computational efficiency.** For problems where the solution is required only at a limited set of time instants (e.g., only the final state or several cross-sections), the hybrid method allows obtaining the result with a single sweep over all frequencies s_k, without step-by-step integration.

The limitations of the method are associated with the need to compute the Laplace transform of the sources and boundary conditions, as well as with the choice of parameters for the numerical inversion. For nonlinear problems, the method is applicable only after linearization.

Comparison with known works [5, 6] shows that our proposal provides a more universal way of handling an arbitrary number of layers with nonhomogeneous sources, while the error of the method is controlled both in space (second order) and in the parameter K.

6. Conclusion

A hybrid numerical-analytical method for solving nonstationary heat and mass transfer problems in multilayered media with discontinuous thermophysical coefficients has been proposed. The method combines finite-difference approximation in space using the harmonic mean at layer interfaces with the integral Laplace transform in time followed by numerical inversion via the Dubner method. This approach avoids step-by-step time integration and the associated stability restrictions characteristic of explicit difference schemes, and also ensures correct reproduction of heat flux conjugation conditions.

It has been proven that for $Re(s) \geq 0$, the system matrix is strictly diagonally dominant, which guarantees the stability and uniqueness of the solution in the transform domain. It has been shown that the total error of the method consists of the spatial component $O(h^2)$ and the exponentially decaying inversion error $O(e^{-cK})$, which at $h = 0.01$ and $K = 100$ provides a relative accuracy of order 10^{-3}.

In test problems (two-layer and three-layer domains), it was established that the hybrid method outperforms the standard implicit finite-difference scheme in accuracy (error 0.9% versus 1.5% in the two-layer test and 1.2% versus 4.1% in the three-layer test with a local source) with a gain in computation time of about 25%. The advantage of the method is most noticeable in reproducing derivative kinks at layer interfaces.

The method is applicable for calculating temperature fields in multilayer structures, biological tissues, and semiconductor structures. Prospects for further research include generalization to multidimensional problems using the finite element method, adaptive selection of Laplace inversion parameters, and solution of inverse coefficient problems.

References

[1] Tikhonov A.N., Samarskii A.A. Equations of Mathematical Physics. Moscow State University Press, 1999. 798 p.

[2] Pennes H.H. Analysis of tissue and arterial blood temperatures in the resting human forearm. *Journal of Applied Physiology*, 1948, vol. 1, no. 2, pp. 93-122.

[3] Lasance C.J.M. The thermal conductivity of silicon dioxide. *Electronics Cooling*, 2006, vol. 12, no. 1, pp. 20-24.

[4] Samarskii A.A. Theory of Difference Schemes. Moscow: Nauka, 1989. 616 p.

[5] Kuznetsov A.V., Nield D.A. Forced convection in a channel partly filled with a porous medium: an exact solution. *International Journal of Heat and Mass Transfer*, 2009, vol. 52, pp. 5288-5292.

[6] Ramm A.G. A numerical method for solving the heat equation in multilayered media. *Mathematics and Computers in Simulation*, 2005, vol. 68, pp. 159-167.

[7] Krylov V.I., Skoblya N.S. A numerical method for the inversion of the Laplace transform. *Differential Equations*, 1969, vol. 5, pp. 158-165.

[8] Samarskii A.A., Vabishchevich P.N. Computational Heat Transfer. “ Wiley, 1995. 432 p.

[9] Dubner H., Abate J. Numerical inversion of Laplace transforms by relating them to the finite Fourier cosine transform. *Journal of the ACM*, 1968, vol. 15, no. 1, pp. 115-123.

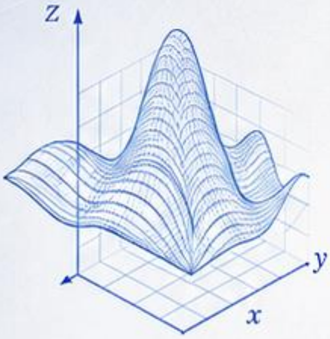
[10] Γ–zisik M.N. Heat Conduction. “ Wiley, 1993. 712 p.

[11] Abdullaev A.A., Khayotov A.R. Numerical solution of nonstationary heat conduction problems in multilayered media. *Uzbek Mathematical Journal*, 2022, no. 4, pp. 12-24.

[12] Hasanov A., Jiwari R. A new hybrid method for solving heat transfer problems in layered materials. *Malaysian Journal of Mathematical Sciences*, 2023, vol. 17, no. 2, pp. 245-260.

JMMME

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING



$$\frac{dx}{dt} = Ax + b(t)$$

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u + f$$

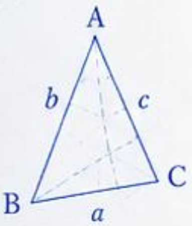
$$\sum_{i=1}^{n} a_i x_i + b$$

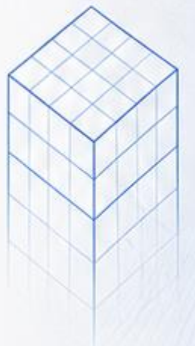
$$f(x) = \int_a^b f(x)\,dx$$

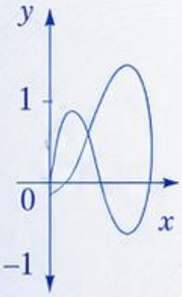
CONTENTS

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\sum_{i=1}^{n} w_i x_i$$







TIIAME National Research University