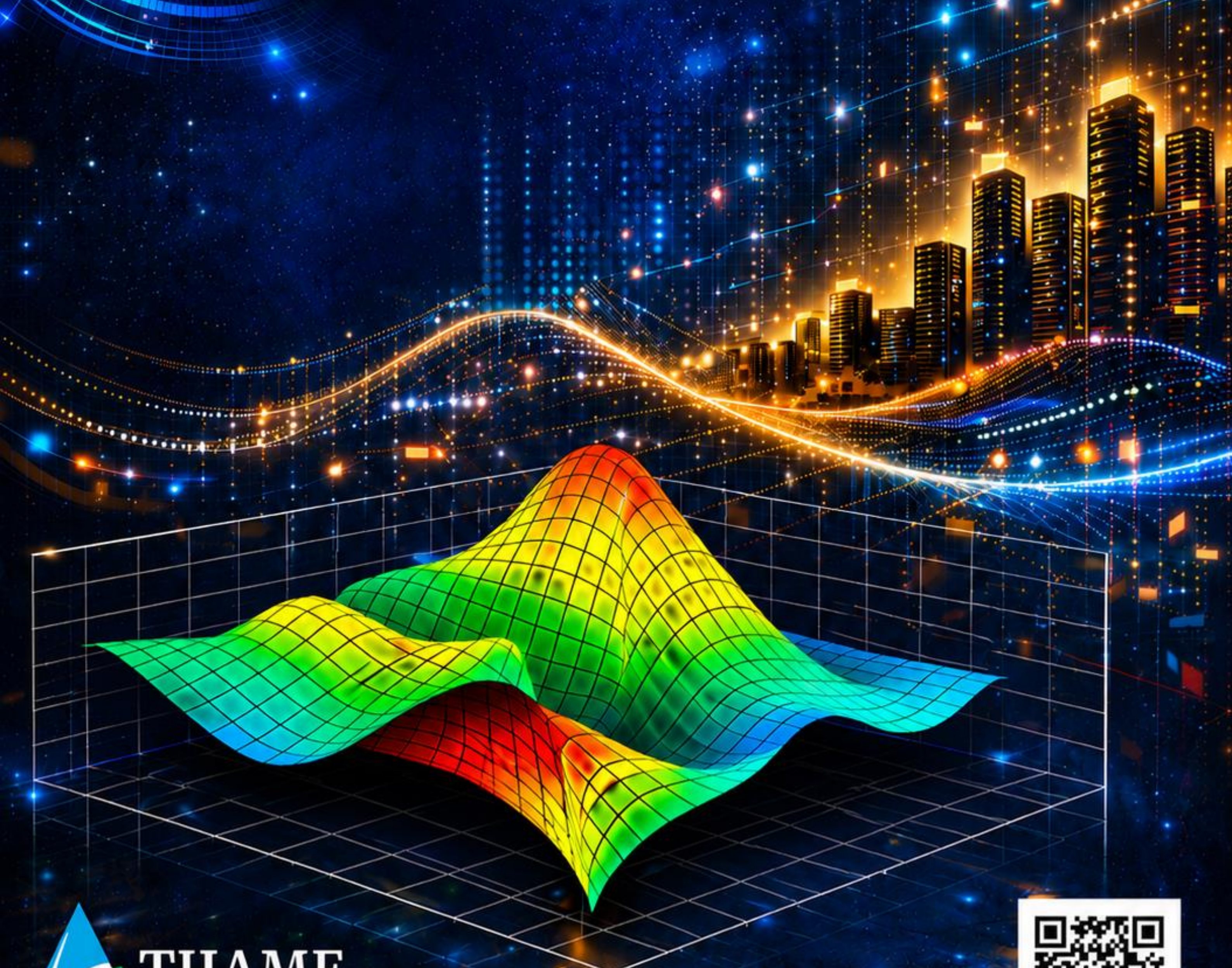
JMMME

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING



TIIAME
National Research University

JMMME

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING

JOURNAL INFORMATION

"Journal of Mathematical Models and Methods in Engineering" is an international, peer-reviewed, and open-access scientific journal established in the Republic of Uzbekistan by the National Research University "Tashkent Institute of Irrigation and Agricultural Mechanization Engineers" under Certificate No. 1648073 dated June 15, 2026.

The journal specializes in publishing scientific research on mathematical modeling, numerical methods, computational algorithms, optimization, and analytical approaches across various fields of engineering — including civil engineering, mechanical engineering, energy, hydraulic engineering, materials science, robotics, and environmental protection. It serves as a scientific platform for scholars, engineers, educators, and doctoral researchers to exchange theoretical and applied knowledge.

All submitted manuscripts undergo a double-blind peer review process and are published in strict compliance with high academic quality standards and publication ethics.

The journal has been published since June 15, 2026, once per quarter, that is, 4 times per year. Manuscripts are accepted in English and must be prepared in accordance with international IMRAD standards.

TIIAME
National Research University

FOUNDER	TIIAME National Research University
ADDRESS	39 Qori-Niyoziy Street, Mirzo Ulugbek District, Tashkent, Republic of Uzbekistan
PHONE	+998 (93) 397-12-39
WEBSITE	jmmme.uz
CERTIFICATE	No. 1648073, June 15, 2026
PUBLISHED	2026-06-15
LANGUAGE	English
FREQUENCY	Quarterly / 4 times per year
REVIEW	Double-blind peer review
STANDARD	International IMRAD standards

INDEXED AND DISCOVERABLE THROUGH

ISSN INTERNATIONAL STANDARD SERIAL NUMBER · Google Scholar · zenodo · doi · OpenAIRE · OPEN ACCESS

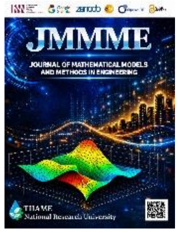
JMMME

TIIAME National Research University

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING

Journal of Mathematical Models and Methods in Engineering

journal homepage: https://jmmme.uz/index.php/journal



Reduction formulas for hypergeometric functions and their application to the comparison of problems solutions for the Laplace equation and singular elliptic equations

M.O.Abbasova[1], T.G.Ergashev[2*], A.R. Ryskan[3]

[1] *Namangan State University, Namangan, 160107, Uzbekistan*
[2] *"Tashkent institute of irrigation and agricultural mechanization engineers" National Research University, Tashkent, 100000, Uzbekistan*
[3] *University of California, 1 Shields Ave, Davis, CA 95616, USA*

ARTICLE INFO

Keywords:
hypergeometric functions Gauss,
Appell and Lauricella,
reduction formulas,
Dirichlet problem,
Neumann problem,
Laplace equation,
elliptic equation with singular coefficients.

ABSTRACT

The main goal of this work is to prove a number of reduction formulas for the first Lauricella function, i.e. to determine the set of those numerical parameters for which the hypergeometric function is expressed in elementary functions. Currently, tables of partial values for Appell functions of two variables are known; however, these formulas are insufficient for applications, and therefore, it is necessary to establish more general relations for hypergeometric functions whose number of variables exceeds 2. In the first part of the work, we will prove new reduction formulas for the Lauricella function of three or more variables, generalizing previously known results. The second part of the paper is devoted to applying the obtained reduction formulas to the study of the properties of solutions to the Dirichlet and Neumann problems for the Laplace equation and a singular elliptic equation. The main result of this study is that, using reduction formulas for hypergeometric functions, it is proven that the solutions to the Dirichlet and Neumann problems for elliptic equations with vanishing singular coefficients coincide with the solutions to similar problems for the Laplace equation.

1. Introduction

It is well known [1,2] that the solution of a wide variety of problems related to heat conduction and dynamics, electromagnetic oscillations and aerodynamics, quantum mechanics, and potential theory leads to special functions. They most often arise when solving partial differential equations using separation of variables or the Green's function method. The diversity of problems leading to special functions has led to a rapid increase in the number of functions used in applications.

The great success of the theory of hypergeometric series in one variable has stimulated the development of a corresponding theory in two and more variables. Appell [3] has defined, in 1880, four double series F_1 to F_4, which are all analogous to Gauss' $F(a,b;c;z)$. Following the Appell's work, Lauricella [4] has defined, in 1893, four hypergeometric series $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$

* Corresponding author. **T.G.Ergashev**
E-mail addresses: ergashev.tukhtasin@gmail.com

and $F_D^{(n)}$, which are natural generalizations of the classical Gauss hypergeometric function F and the Appell functions $F_1 - F_4$ to the case in n complex variables and the corresponding complex parameters.

The Appell and Lauricella hypergeometric functions play an important role in mathematical physics. In particular, solutions to a number of boundary value problems for singular elliptic equations are given in terms of the Appell hypergeometric function F_2 [5,6] and the Lauricella hypergeometric function $F_A^{(n)}$ [7,8,9,10]. Under exceptional circumstances, multiple hypergeometric functions can be expressed in terms of simpler functions, notably in terms of hypergeometric functions of one variable or, in general, with fewer variables or in terms of elementary functions. In such cases we speak of *reducible* hypergeometric functions and of *reduction formulas*. Certain trivial reduction formulas are obvious: for example, if one of the two variables is zero in any of the double series, the hypergeometric series in two variables can be expressed in terms of series in one variable: such trivial reductions are disregarded in the sequel.

The monographs [11, 12, 13] contain a systematic exposition of summation, reduction, and transformation formulas, as well as various other developments, covering not only each of the four Appell functions, but also several families of functions in two variables. Furthermore, the works [14] and [15] compiled an extensive tables of reduction formulas for the double Appell F_2 and triple Lauricella $F_A^{(3)}$ functions, respectively.

In this paper, for the first time, reduction formulas are established for the Lauricella hypergeometric function $F_A^{(n)}$ in n variables and the results obtained are applied to the study of the properties of solutions of the Dirichlet and Neumann problems for the Laplace equation and the elliptic equation with singular coefficients in the infinite parts (half-space, quarter-space, half-quarter-space, in general, 1/2 -part of space) of m-dimensional Euclidean space.

2.Reduction formulas for Gauss and Appell functions at zero parameter values

The Gauss hypergeometric function is defined inside the circle $|z|<1$ as the sum of the hypergeometric series [12, p. 109, eq. 2.8(1)]

$$F(a,b;c;z) \equiv F\left(\begin{matrix} a,b; \\ c; \end{matrix} z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1}$$

and for $|z|>1$, as its analytic continuation. Here $a,b \in \mathbb{C}$ и $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\mathbb{Z}_0^{-1} := \{0,-1,-2,...\}$, and the expression $(\nu)_n$ denotes the Pochhammer symbol: $(\nu)_0 := 1,\ (\nu)_n := \nu(\nu+1)...(\nu+n-1)$, $\nu \in \mathbb{C}$, $n \in \mathbb{N}$, written through the gamma function $\Gamma(\nu)$ in the form $(\nu)_n = \Gamma(\nu+n)/\Gamma(\nu),\ n \in \{0\} \cup \mathbb{N}$; $\mathbb{N}$ is the set of natural numbers.

In general, an extensive table is known [16, Chapter 7, Section 7.3], in which the hypergeometric function for certain particular values of numerical parameters is written in terms of elementary functions, for example,

$$F\left(a+1,1;2;z\right) = \frac{1}{az}\left[(1-z)^{-a} - 1\right], a \neq 0, \tag{2}$$

$$F\left(1,1;2;z\right) = -\frac{1}{z}\ln(1-z). \tag{3}$$

The Appell hypergeometric function F_2 has a form [12, p. 224, eq. 5.7(7)]

$$F_2\left(a,b,b';c,c';x,y\right) \equiv F_2\left(\begin{matrix} a,b,b'; \\ c,c'; \end{matrix} x,y\right)$$
$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\left(b\right)_m\left(b'\right)_n}{\left(c\right)_m\left(c'\right)_n} \frac{x^m}{m!}\frac{y^n}{n!}, |x|+|y|<1, \tag{4}$$

where $a,b,b' \in \mathbb{C}$ and $c,c' \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

J.Murley and N.Saad [14] compiled an extensive tables of the Appell functions F_2. Let us list some of them that we need to continue our research:

$$F_2\left(\begin{matrix} a+2,1,1; \\ 2,2; \end{matrix} x,y\right) = \frac{1}{a(a+1)xy}\left[1-\frac{1}{(1-x)^a}-\frac{1}{(1-y)^a}+\frac{1}{(1-x-y)^a}\right], a\neq 0,-1, \tag{5}$$

$$F_2\left(2,1,1;2,2;x,y\right) = \frac{1}{xy}\ln\frac{(1-x)(1-y)}{1-x-y} \equiv \frac{1}{xy}\ln\left(1+\frac{xy}{1-x-y}\right). \tag{6}$$

However, the table [14] of Appell functions $F_2\left(a,b,1;c,2;x,y\right)$ cannot claim to be complete in this direction, because in applications of the Appell function F_2 it is necessary to calculate the values of this function, i.e. to express it in elementary functions, when some of its parameters, which are in the numerator and denominator, simultaneously tend to zero.

Hypergeometric functions can be expressed through elementary functions when the same number of parameters of the numerator and denominator are simultaneously equal to zero.

The following reduction formulas are valid:

$$\lim_{b\to 0} F\left(a,b;kb;x\right) = \frac{1}{k}\left[(1-x)^{-a}+k-1\right],\ k\neq 0. \tag{7}$$

$$\lim_{b\to 0} F\left(a+\lambda,b+\lambda;kb+\lambda;x\right) = (1-x)^{-a-\lambda},\ k\neq 0,\ \ \lambda\neq 0. \tag{8}$$

Reduction formulas (5) – (8) can be proved by comparing coefficients of equal powers of variables on both sides.

Theorem 1. *For* $|x|+|y|<1$, *the Appell function* F_2 *is given by*

$$F_2\left(a,b,b';kb,k'b';x,y\right)\Big|_{b=b'=0}$$
$$= \frac{1}{kk'}\left[(k-1)(k'-1)+\frac{k'-1}{(1-x)^a}+\frac{k-1}{(1-y)^a}+\frac{1}{(1-x-y)^a}\right],\ k\neq 0, k'\neq 0; \tag{9}$$

$$F_2\left(a+1,1,b;2,kb;x,y\right)\Big|_{b=0}$$
$$= -\frac{1}{akx}\left[k-1-\frac{k-1}{(1-x)^a}+\frac{1}{(1-y)^a}-\frac{1}{(1-x-y)^a}\right],\ a\neq 0,\ k\neq 0; \tag{10}$$

$$F_2\left(1,1,b;2,kb;x,y\right)\Big|_{b=0} = -\frac{1}{x}\ln(1-x)+\frac{1}{kx}\ln\left(1+\frac{xy}{1-x-y}\right). \tag{11}$$

Proof. The well-known Burchnall-Chaundy expansion [17]

$$F_2\left(\begin{matrix} a,b,b'; \\ c,c'; \end{matrix} x,y\right) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n(b')_n}{n!(c)(c')_n} x^n y^n F\left(\begin{matrix} a+n,b+n; \\ c+n; \end{matrix} x\right) F\left(\begin{matrix} a+n,b'+n; \\ c'+n; \end{matrix} y\right)$$

allows one to easily derive the reduction formula (9). Indeed,

$$
\begin{aligned}
&F_2\left(a,b,b';kb,k'b';x,y\right)=F(a,b;kb;x)F(a,b';k'b';y)\\
&\quad+\frac{a}{k\cdot k'}xyF(a+1,b+1;kb+1;x)F(a+1,b'+1;k'b'+1;y)\\
&\quad+\frac{a(a+1)(b+1)(b'+1)}{2k\cdot k'(kb+1)(k'b'+1)}x^2y^2F(a+2,b+2;kb+2;x)\times\\
&\quad\times F(a+2,b'+2;k'b'+2;y)+.... \qquad (12)
\end{aligned}
$$

Next, passing to the limit in both parts of equality (12) as $b\to 0$ and $b'\to 0,$ by virtue of the formulas (7) and (8), after reducing similar ones, we obtain the reduction formula (9).

Applying the formula (7) to the expansion [16, p. 413, eq. 6.7.1(4)]

$$
F_2\left(a,b,b';c,c';x,y\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b')_n}{n!(c')_n}y^nF(a+n,b;c;x),\quad |x|+|y|<1
$$

for $b'=1,\ c'=2$, and passing to the limit as $b\to 0,$ we obtain the required formula (10).

Formula (11) follows directly from the reduction formula (3). Theorem 1 is completely proven.

3.Reduction formulas for the multiple Lauricella function $F_A^{(n)}$

We introduce a following notation:

$$
\mathbf{b}:=\left(b_1,...,b_n\right),\ \mathbf{c}:=\left(c_1,...,c_n\right),\ \mathbf{x}:=\left(x_1,...,x_n\right);\ |\mathbf{k}|:=k_1+...+k_n,\ k_1\ge 0,...,k_n\ge 0.
$$

The Lauricella hypergeometric function $F_A^{(n)}$ in $n\in N$ (real or complex) variables is defined as [4] (see also [11, p.114])

$$
F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)\equiv F_A^{(n)}\left(\begin{matrix}a,b_1,...,b_n;\\ c_1,...,c_n;\end{matrix}x_1,...,x_n\right)
$$

$$
=\sum_{|\mathbf{k}|=0}^{\infty}(a)_{|\mathbf{k}|}\prod_{j=1}^{n}\left[\frac{(b_j)_{k_j}}{(c_j)_{k_j}}\frac{x_j^{k_j}}{k_j!}\right],\quad \sum_{j=1}^{n}|x_j|<1, \qquad (13)
$$

where $a,b_1,...,b_n\in\mathbb{C}$, $c_1,...,c_n\in\mathbb{C}\setminus\mathbb{Z}_0^-$.

Clearly, in definition (13) we have $F_A^{(1)}\equiv F$ and $F_A^{(2)}\equiv F_2,$ where F and F_2 are Gauss and Appell series defined in (1) and (4), respectively.

Using the properties of the Pochhammer symbol, the Lauricella function $F_A^{(n)}$ can be expanded into hypergeometric functions with fewer variables:

$$
F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right)=\sum_{m_n=0}^{\infty}\frac{(a)_{m_n}\left(b_n\right)_{m_n}}{\left(c_n\right)_{m_n}}\frac{x_n^{m_n}}{m_n!}F_A^{(n-1)}\left(a+m_n,\mathbf{b}';\mathbf{c}';\mathbf{x}'\right) \qquad (14)
$$

$$
=\sum_{|\mathbf{m}'|=0}^{\infty}(a)_{|\mathbf{m}'|}\prod_{j=1}^{n-1}\left[\frac{(b_j)_{m_j}}{(c_j)_{m_j}}\frac{x_j^{m_j}}{m_j!}\right]F\left(a+|\mathbf{m}'|,b_n;c_n;x_n\right), \qquad (15)
$$

where $\mathbf{b}':=\left(b_1,...,b_{n-1}\right),\ \ \mathbf{c}':=\left(c_1,...,c_{n-1}\right),\ \ \mathbf{x}':=\left(x_1,...,x_{n-1}\right),\ |\mathbf{m}'|:=m_1+...+m_{n-1}$

.

Theorem 2. *For* $\sum_{j=1}^{n}|x_j|+|y|<1,$ *the function* $F_A^{(n+1)}$ *in* $n+1$ *variables is given by*

$$F_A^{(n+1)}\left(\begin{matrix} a+1,\mathbf{b},1; \\ \mathbf{c},2; \end{matrix}\mathbf{x},y\right) = -\frac{1}{ay}F_A^{(n)}\left(\begin{matrix} a,\mathbf{b}; \\ \mathbf{c}; \end{matrix}\mathbf{x}\right) + \frac{1}{ay(1-y)^a}F_A^{(n)}\left(\begin{matrix} a,\mathbf{b}; \\ \mathbf{c}; \end{matrix}\frac{\mathbf{x}}{1-y}\right), \quad (16)$$

where $a \neq 0$; $b_1,...,b_n \in \mathbb{C}$, $c_1,...,c_n \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

Proof. Using expansions (14) and (15), the Lauricella function $F_A^{(n+1)}$ in $n+1$ variables can be easily represented in the form

$$F_A^{(n+1)}\left(\begin{matrix} a+1,\mathbf{b},1; \\ \mathbf{c},2; \end{matrix}\mathbf{x},y\right) = \sum_{|\mathbf{k}|=0}^{\infty}(a+1)_{|\mathbf{k}|}\prod_{j=1}^{n}\left[\frac{(b_j)_{k_j}}{(c_j)_{k_j}}\frac{x_j^{k_j}}{k_j!}\right]F\left(a+1+|\mathbf{k}|,1;2;y\right). \quad (17)$$

Applying the well-known reduction formula

$$F\left(a+1,1;2;z\right) = \frac{1}{az}\left[(1-z)^{-a}-1\right], a \neq 0,$$

to the right-hand side of (17), we obtain

$$F_A^{(n+1)}\left(\begin{matrix} a+1,\mathbf{b},1; \\ \mathbf{c},2; \end{matrix}\mathbf{x},y\right) = \frac{1}{y}\sum_{|\mathbf{k}|=0}^{\infty}\frac{(a+1)_{|k|}}{a+|\mathbf{k}|}\prod_{j=1}^{n}\left[\frac{(b_j)_{k_j}}{(c_j)_{k_j}}\frac{x_j^{k_j}}{k_j!}\right]\left[(1-y)^{-a-|k|}-1\right].$$

The last equality directly implies equality (16). Theorem 2 is proved.

Corollary 1. *If* $a \neq 0,-1,...,-(n-1)$; $b \in \mathbb{C}$, $c \in \mathbb{C}\setminus\mathbb{Z}_0^-$, *then the following reduction formulas*

$$F_A^{(n+1)}\left(\begin{matrix} a+n,\underbrace{1,...,1}_{n},b; \\ \underbrace{2,...,2}_{n},c; \end{matrix}\mathbf{x},y\right) = \frac{(-1)^n}{(a)_n\Pi\left(x_n\right)}\sum_{I_s\subseteq\{1,...,n\}}\frac{(-1)^{|I_n|}}{\left(1-X_{I_n}\right)^a}F\left(a,b;c;\frac{y}{1-X_{I_s}}\right), \quad (18)$$

$$F_A^{(n)}\left(a+s,\underbrace{1,...,1}_{s},b_{s+1},...,b_n;\underbrace{2,...,2}_{s},c_{s+1},...,c_n;x_1,...,x_n\right) = \frac{(-1)^s}{(a)_s\Pi\left(x_s\right)}\times$$

$$\times\sum_{I_s\subseteq\{1,...,s\}}\frac{(-1)^{|I_s|}}{\left(1-X_{I_s}\right)^a}F_A^{(n-s)}\left(a,b_{s+1},...,b_n;c_{s+1},...,c_n;\frac{x_{s+1}}{1-X_{I_s}},...,\frac{x_n}{1-X_{I_s}}\right) \quad (19)$$

are valid.

Here and below, I_n denotes any subset of the set of first n natural numbers $\{1,2,...,n\}$, i.e., the right-hand sides of the reduction formulas (18) and (19) consist of 2^n and 2^s pieces of terms, respectively; $X_{I_n} = \sum_{i\in I_n}x_i$, $X_\varnothing = 0$; $|I_n|$ is an element's number of set I_n, $|\varnothing| = 0$; $\Pi\left(x_k\right)$ is a product of the first k variables of the Lauricella function $\left(1 \leq k \leq n\right)$.

Examples.

$$F_A^{(n)}\left(a+n,\underbrace{1,...,1}_{n};\underbrace{2,...,2}_{n};\mathbf{x}\right) = \frac{(-1)^n}{(a)_n\Pi\left(x_n\right)}\sum_{I_n\subseteq\{1,...,n\}}\frac{(-1)^{|I_n|}}{\left(1-X_{I_n}\right)^a}, \quad (20)$$

$$F_A^{(n)}\left(n,\underbrace{1,...,1}_{n};\underbrace{2,...,2}_{n};\mathbf{x}\right) = \frac{(-1)^n}{(n-1)!\Pi\left(x_n\right)}\sum_{I_n\subseteq\{1,...,n\}}(-1)^{|I_n|}\ln\frac{1}{1-X_{I_n}}. \quad (21)$$

It should be noted that formulas (20) and (21) are natural generalizations of the reduction formulas (2), (5) and (3), (6), respectively.

Theorem 3. *For* $\sum_{j=1}^{n}|x_j|<1,$ *and* $k_1\neq 0$,..., $k_n\neq 0,$ *the function* $F_A^{(n)}$ *is given by*

$$F_A^{(n)}\left(\begin{matrix}a,b_1,...,b_n;\\ k_1b_1,...,k_nb_n;\end{matrix}x_1,...,x_n\right)\Bigg|_{\mathbf{b}=0}=\frac{1}{k_1k_2...k_n}\sum_{I_n}\frac{1}{\left(1-X_{I_n}\right)^a}. \tag{22}$$

Proof. We prove the reduction formula (22) by mathematical induction. Formula (22) is true for $n=1$ and $n=2$ (see eq. (7) and (9), respectively).

Let us assume that formula (22) is valid for any natural number n. We will prove that it is also true for $n\sim n+1$. We write the expansion (14) in the form

$$F_A^{(n+1)}\left(\begin{matrix}a,b_1,...,b_n,b_{n+1};\\ k_1b_1,...,k_nb_n,k_{n+1}b_{n+1};\end{matrix}x,x_{n+1}\right)$$

$$=\sum_{s=0}^{\infty}\frac{(a)_s\left(b_{n+1}\right)_s}{\left(k_{n+1}b_{n+1}\right)_s}\frac{x_{n+1}^s}{s!}F_A^{(n)}\left(\begin{matrix}a+s,b_1,...,b_n;\\ k_1b_1,...,k_nb_n;\end{matrix}x\right)$$

$$=F_A^{(n)}\left(\begin{matrix}a,b_1,...,b_n;\\ k_1b_1,...,k_nb_n;\end{matrix}x\right)+\frac{1}{k_{n+1}}\sum_{s=1}^{\infty}\frac{(a)_s\left(1+b_{n+1}\right)_{s-1}}{\left(1+k_{n+1}b_{n+1}\right)_{s-1}}\frac{x_{n+1}^s}{s!}F_A^{(n)}\left(\begin{matrix}a+s,b_1,...,b_n;\\ k_1b_1,...,k_nb_n;\end{matrix}x\right).$$

Passing to the limit in the last expression as $\mathbf{b}\to 0$ and $b_{n+1}\to 0$, by virtue of (22), we obtain

$$F_A^{(n+1)}\left(\begin{matrix}a,b_1,...,b_n,b_{n+1};\\ k_1b_1,...,k_nb_n,k_{n+1}b_{n+1};\end{matrix}x,x_{n+1}\right)\Bigg|_{\mathbf{b}=0,b_{n+1}=0}$$

$$=\frac{1}{k_1k_2...k_n}\sum_{I_n\subseteq\{1,...,n\}}\frac{1}{\left(1-X_{I_n}\right)^a}\left[k_{n+1}+\sum_{s=1}^{\infty}\frac{(a)_s}{s!}\left(\frac{x_{n+1}}{1-X_{I_n}}\right)^s\right].$$

Now, taking into account the easily verified equality

$$\sum_{s=1}^{\infty}\frac{(a)_s}{s!}\left(\frac{x_{n+1}}{1-X_I}\right)^s=\frac{\left(1-X_{I_n}\right)^a}{\left(1-X_{I_n}-x_{n+1}\right)^a}-1,$$

after several transformations, we have

$$F_A^{(n+1)}\left(\begin{matrix}a,b_1,...,b_n,b_{n+1};\\ k_1b_1,...,k_nb_n,k_{n+1}b_{n+1};\end{matrix}x,x_{n+1}\right)\Bigg|_{b=0,b_{n+1}=0}=\frac{1}{k_1k_2...k_nk_{n+1}}\sum_{I_{n+1}\subseteq\{1,...,n+1\}}\frac{1}{\left(1-X_{I_{n+1}}\right)^a},$$

which is what was required to be proved. Theorem 3 is proven.

Theorem 4. *Let* s *be a natural number* ($1\leq s\leq n$) *and* n *be the number of variables of the Lauricella function* $F_A^{(n)}$. *For* $\sum_{j=1}^{n}|x_j|<1,$ *and* $k_{s+1}\neq 0$,..., $k_n\neq 0,$ *the function* $F_A^{(n)}$ *is given by*

$$F_A^{(n)}\left(a+s,\underbrace{1,...,1}_{s},b_{s+1},...,b_n;\underbrace{2,...,2}_{s},k_{s+1}b_{s+1},...,k_nb_n;x_1,...,x_n\right)\Bigg|_{\substack{b_{s+1}=0\\ \cdots \\ b_n=0}}$$

$$=\frac{(-1)^s}{(a)_s\Pi(x_s)k_{s+1}...k_n}\sum_{I_s\subseteq\{1,...,s\}}(-1)^{|I|}\sum_{J_n\subseteq\{s+1,...,n\}}\frac{1}{\left(1-X_{I_s}-X_{J_n}\right)^a}. \quad (23)$$

Proof. Reduction formula (23) is proved by mathematical induction, similar to the proof of the reduction formula (22).

4.Applications of the obtained results

Let $\Re_m$ be m-multidimensional Euclidean space ($m\ge 2$), $x:=(x_1,...,x_m)$ be an arbitrary point in it, n be a natural number, where $n\le m.$ We define the 2^n th part of the Euclidean space $\Re_m$ as follows:

$$\Omega\equiv\Omega_m^{n+}=\left\{x\in\Re_m: x_i>0,\ i=1,...,n;-\infty<x_j<+\infty,\ j=n+1,...,m\right\}.$$

We consider the Laplace equation

$$\Delta u\equiv\sum_{i=1}^{m}\frac{\partial^2 u}{\partial x_i^2}=0 \quad (24)$$

and generalized singular elliptic equation

$$L_\alpha^{(m,n)}(u)\equiv\sum_{i=1}^{m}\frac{\partial^2 u}{\partial x_i^2}+\sum_{j=1}^{n}\frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j}=0 \quad (25)$$

in $\Omega,$ where $\alpha:=(\alpha_1,...,\alpha_n)$ is a vector and α_j are real numbers: $0<2\alpha_j<1,\ j=1,...,n.$

In the case $m=2$ the detailed bibliography and exposition of studies of basic boundary value problems for equations (14) and (25) can be found un the monographs [18] – [20].

In this paper we set $m>2.$

We denote:

$$S_k=\left\{\begin{matrix}x: x_i>0,\ ...,x_{k-1}>0,\ x_k=0,x_{k+1}>0,...,x_n>0;\\ -\infty<x_j<+\infty,\ j=n+1,...,m\end{matrix}\right\};$$

$$\tilde{x}_k:=(x_1,...,x_{k-1},x_{k+1},...,x_n,...,x_m)\in S_k\subset\Re_{m-1};\quad \rho_k:=\sqrt{\textstyle\sum_{i=1,i\ne k}^{m}x_k^2};$$

$$\xi:=(\xi_1,...,\xi_m)\in\Re_m;\ \tilde{\xi}_k:=(\xi_1,...,\xi_{k-1},\xi_{k+1},...,\xi_n,...,\xi_m)\in S_k\subset\Re_{m-1};$$

$$r=\sqrt{\sum_{j=1}^{m}\left(x_j-\xi_j\right)^2};\ r_{I_n}=\sqrt{\sum_{j\in I_n}\left(x_j+\xi_j\right)^2+\sum_{j\in\{1,...,m\}\backslash I_n}\left(x_j-\xi_j\right)^2};\ r_\varnothing\equiv r.$$

Dirichlet problem $D_{m,n}^\infty$. Find a solution $u(x)\in C(\overline{\Omega})\cap C^2(\Omega)$ to the Laplace equation (to the singular equation (25)), satisfying the conditions

$$u(x)\Big|_{x_k=0}=\tau_k(\tilde{x}_k),\quad \tilde{x}_k\in\overline{S}_k,\quad k=\overline{1,n},\quad \lim_{R\to\infty}u(x)=0,$$

where $\tau_k(\tilde{x}_k)$ are given functions such that $\tau_k(\tilde{x}_k)=O\left(\theta_k^{-\varepsilon}\right),\ \theta_k\to\infty,\ \varepsilon>0,k=\overline{1,n}.$

In addition, the functions $\tau_k(\tilde{x}_k)$ satisfy the concordance conditions

$$\tau_k(\mathbf{0})=\tau_j(\mathbf{0}),\ \tau_k(\tilde{x}_k)\Big|_{x_j=0}=\tau_j(\tilde{x}_j)\Big|_{x_k=0},\ \ k\neq j,\ k,j=\overline{1,n},\ \mathbf{0}:=(0,...,0)\in\Re_{m-1}.$$

The solution $u_0^{(n)}(x)$ of the Dirichlet problem $D_{m,n}^{\infty}$ for the Laplace equation (24) in Ω has a form [21]

$$u_0^{(n)}(x)=\frac{\Gamma(m/2)}{\pi^{m/2}}\sum_{k=1}^{n}x_k\underbrace{\int_0^{\infty}...\int_0^{\infty}}_{n-1}\underbrace{\int_{-\infty}^{\infty}...\int_{\infty}^{\infty}}_{m-n}\tau_k\left(\tilde{\xi}_k\right)\sum_{I_{n\backslash k}}\frac{(-1)^{|I_{n\backslash k}|}}{r_{I_{n\backslash k}}^{m}}d\tilde{\xi}_k, \tag{26}$$

and the solution $u_\alpha^{(n)}(x)$ of the Dirichlet problem $D_{m,n}^{\infty}$ for the singular elliptic equation (25) in Ω is also known [8]:

$$u_\alpha^{(n)}(x)=\sum_{k=1}^{n}\underbrace{\int_0^{\infty}...\int_0^{\infty}}_{n-1}\underbrace{\int_{-\infty}^{\infty}...\int^{\infty}}_{m-n}\tau_k\left(\tilde{\xi}_k\right)T_{\alpha,k}^{(n)}\left(x;\tilde{\xi}_k\right)d\tilde{\xi}_p, \tag{27}$$

where

$$T_{\alpha,k}^{(n)}\left(x,\tilde{\xi}_k\right)=\ \left(1-2\alpha_k\right)\gamma_n x^{(1-2\alpha)}\tilde{\xi}_k^{(1)}\times$$
$$\times R_k^{-2\beta_n}F_A^{(n-1)}\left(\begin{matrix}\beta_n,1-\alpha_1,...,1-\alpha_{k-1},1-\alpha_{k+1},...,1-\alpha_n;\\ 2-2\alpha_1,...,2-2\alpha_{k-1},2-2\alpha_{k+1},...,2-2\alpha_n;\end{matrix}\tilde{\rho}_k\right),$$

$$\tilde{\rho}_k:=\left(\rho_1,...,\rho_{k-1},\rho_{k+1},...,\rho_n\right),\quad \rho_j=-\frac{4x_j\xi_j}{R_k^2},\ j\neq k,\ j=\overline{1,n},$$

$$\beta_n=\frac{m-2}{2}+n-\sum_{j=1}^{n}\alpha_j,\quad \gamma_n=2^{2\beta_n-m}\frac{\Gamma(\beta_n)}{\pi^{m/2}}\prod_{j=1}^{n}\left[\frac{\Gamma\left(1-\alpha_j\right)}{\Gamma\left(2-2\alpha_j\right)}\right];$$

$$x^{(1-2\alpha)}=\prod_{j=1}^{n}\left[x_j^{1-2\alpha_j}\right],\quad \tilde{\xi}_k^{(1)}=\prod_{j=1,\,j\neq k}^{n}\left[\xi_k\right],\ 1\leq k\leq n.$$

Theorem 5. *Let the functions* $u_0^{(n)}(x)$ *and* $u_\alpha^{(n)}(x)$*, defined by equalities (26) and (27), be solutions of the Dirichlet problems* $D_{m,n}^{\infty}$ *for the Laplace equation (24) and the singular elliptic equation (25), respectively. Then the limit relation*

$$\lim_{\alpha\to 0}u_\alpha^{(n)}(x)=u_0^{(n)}(x) \tag{28}$$

is valid.

Proof. To prove the validity of the limit relation (28), we use the reduction formula (20) for $a\sim m/2$ and $n\sim n-1$:

$$F_A^{(n-1)}\left(\begin{matrix}\frac{m}{2}+n-1,\underbrace{1,...,1}_{n-1};\\ \underbrace{2,...,2}_{n-1};\end{matrix}\tilde{\rho}_k\right)=\frac{(-1)^{n-1}}{(m/2)_{n-1}\Pi_{n-1}\left(\tilde{\rho}_k\right)}\sum_{I_n\subseteq\{1,...,n\}\backslash\{k\}}\frac{(-1)^{|I_n|}}{\left(1-X_{I_n}\right)^{m/2}}. \tag{29}$$

Now, passing to the limit in $J_{\alpha,k}^{(n)}\left(x,\tilde{\xi}_k\right)$ as $\alpha \to 0$ and substituting (29) into $\lim\limits_{\alpha\to 0} u_\alpha^{(n)}(x)$, we obtain the solution (26) to the Dirichlet problem $D_{m,n}^{\infty}$ for the Laplace equation (24). Theorem 5 is proven.

Let $D_n = \bigcup\limits_{k=1}^{n} S_k$ be the union of all n lateral faces of Ω.

Neumann problem $N_{m,n}^{0,\infty}$. Find a solution $u(x) \in C\left(\overline{\Omega}\right) \cap C^1\left(\Omega \cup D_n\right) \cap C^2(\Omega)$ to the Laplace equation (24), satisfying the conditions

$$\left.\frac{\partial u}{\partial x_k}\right|_{x_k=0} = \nu_k(\tilde{x}_k),\ \ \tilde{x}_k \in S_k,\ \ k=\overline{1,n},\quad \lim_{R\to\infty} u(x) = 0,$$

where $\nu_k\left(\tilde{x}_k\right)$ are given functions such that at the origin they can become infinite of integrable order, i.e. less than $m-1$, and $\nu_k\left(\tilde{x}_k\right) = O\left(\theta_k^{-\varepsilon-1}\right),\ \theta_k \to \infty,\ \varepsilon > 0,\ k=\overline{1,n}.$

Neumann problem $N_{m,n}^{\alpha,\infty}$. Find a solution $u(x) \in C\left(\overline{\Omega}\right) \cap C^1\left(\Omega \cup D_n\right) \cap C^2(\Omega)$ to the singular elliptic equation (25), satisfying the conditions

$$\left.\left(x_k^{2\alpha_k}\frac{\partial u}{\partial x_k}\right)\right|_{x_k=0} = \nu_k(\tilde{x}_k),\ \ \tilde{x}_k \in S_k,\ \ k=\overline{1,n},\quad \lim_{R\to\infty} u(x) = 0,$$

where $\nu_k\left(\tilde{x}_k\right)$ are given functions such that at the origin they can become infinite of integrable order, i.e. less than $m-1-2\alpha_k$, and $\nu_k\left(\tilde{x}_k\right) = O\left(\theta_k^{2\alpha_k-\varepsilon-1}\right),\ \theta_k \to \infty,\ \varepsilon > 0,\ k=\overline{1,n}.$

The solution $\tilde{u}_0^{(n)}(x)$ of the Neumann problem $N_{m,n}^{0,\infty}$ for the Laplace equation (24) in Ω has a form [22]

$$\tilde{u}_0^{(n)}(x) = -\frac{1}{2\pi^{m/2}}\Gamma\left(\frac{m-2}{2}\right)\sum_{k=1}^{n}\underbrace{\int_0^{\infty}\ldots\int_0^{\infty}}_{n-1}\underbrace{\int_{-\infty}^{\infty}\ldots\int_{\infty}^{\infty}}_{m-n}\nu_k\left(\tilde{\xi}_k\right)\sum_{I_{n\setminus k}}\frac{1}{r_{I_{n\setminus k}}^{m-2}}d\tilde{\xi}_k, \tag{30}$$

and the solution $\tilde{u}_\alpha^{(n)}(x)$ of the Neumann problem $N_{m,n}^{\alpha,\infty}$ for the singular elliptic equation (25) in Ω is also known [9]:

$$\tilde{u}_\alpha^{(n)}(x) = -\sum_{k=1}^{n}\underbrace{\int_0^{\infty}\ldots\int_0^{\infty}}_{n-1}\underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{m-n}\nu_k\left(\tilde{\xi}_k\right)N_{\alpha,k}^{(n)}\left(x;\tilde{\xi}_k\right)d\tilde{\xi}_k, \tag{31}$$

where $$N_{\alpha,k}^{(n)}\left(x,\tilde{\xi}_k\right) = \gamma_0 R_k^{-2\beta_0} F_A^{(n-1)}\left(\begin{matrix}\beta_0,\alpha_1,\ldots,\alpha_{k-1},\alpha_{k+1},\ldots,\alpha_n;\\ 2\alpha_1,\ldots,2\alpha_{k-1},2\alpha_{k+1},\ldots,2\alpha_n;\end{matrix}\tilde{\rho}_k\right),$$

$$\beta_0 = \frac{m-2}{2} + \sum_{j=1}^{n}\alpha_j,\quad \gamma_0 = 2^{2\beta_0-m}\frac{\Gamma\left(\beta_0\right)}{\pi^{m/2}}\prod_{j=1}^{n}\left[\frac{\Gamma\left(\alpha_j\right)}{\Gamma\left(2\alpha_j\right)}\right].$$

Theorem 6. *Let the functions* $\tilde{u}_0^{(n)}(x)$ *and* $\tilde{u}_\alpha^{(n)}(x)$, *defined by equalities (30) and (31), be solutions of the Neumann problems* $N_{m,n}^{0,\infty}$ *and* $N_{m,n}^{\alpha,\infty}$ *for the Laplace equation (24) and the singular elliptic equation (25), respectively. Then the limit relation*

$$\lim_{\alpha\to 0}\tilde{u}_\alpha^{(n)}(x)=\tilde{u}_0^{(n)}(x) \tag{32}$$

is valid.

Proof. To prove the validity of the limit relation (32), we use the reduction formula (20) for $a\sim(m-2)/2$, $n\sim n-1$ and $k_1=...=k_n=2$:

$$F_A^{(n-1)}\left(\begin{matrix}\dfrac{m-2}{2},\alpha_1,...,\alpha_{k-1},\alpha_{k+1},...,\alpha_n;\\ 2\alpha_1,...,2\alpha_{k-1},2\alpha_{k+1},...,2\alpha_n;\end{matrix}x_1,...,x_{k-1},x_{k+1},...,x_n\right)\Bigg|_{\tilde{\alpha}_k=0}=\frac{1}{2^{n-1}}\sum_{I_{n\backslash k}}\frac{1}{\left(1-X_{I_{n\backslash k}}\right)^{(m-2)/2}},$$

where $\tilde{\alpha}_k:=\left(\alpha_1,...,\alpha_{k-1},\alpha_{k+1},...,\alpha_n\right)$.

5. Conclusions

1. The well-known tables for the Appell hypergeometric function are supplemented with new reduction formulas at zero values of Appell function.
2. For the first time, reduction formulas for the Lauricella hypergeometric function in three or more variables were established
3. The proved reduction formulas for hypergeometric functions were applied to the comparison of solutions of the Dirichlet and Neumann problems for the Laplace equations and singular elliptic equations in infinite parts of Euclidean space.

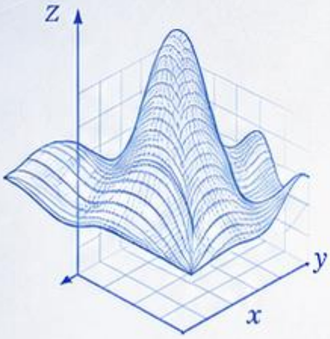
References

[1] L.Bers, Mathematical aspects of subsonic and transonic gas dynamics. New York, Wiley, 1961.
[2] A.W.Niukkanen, Generalised hypergeometric series arising in physical and quantum chemical applications, Journal of Physics. A: Math. Gen., 16 (1983) 1813 – 1825. DOI:10.1088/0305-4470/16/9/007
[3] Appell P. Sur les séries hypergéométriques de deux variables, et sur des équations différentielles linéaires aux dérivées partielles. C.R. Acad. Sci., Paris, 90 (1880) 296 – 298.
[4] G.Lauricella, Sulle funzione ipergeometriche a più variabili, Rend.Circ. Mat. Palermo, 7 (1893) 111 – 158. DOI:10.1007/BF03012437
[5] E.T.Karimov, On a boundary problem with Neumann's condition for 3D singular elliptic equations, Applied Mathematics Letters, 23 (2010) 517 – 522. DOI: 10.1016/j.aml.2010.01.002
[6] E.T.Karimov, J.J.Nieto, The Dirichlet problem for a 3D elliptic equation with two singular coefficients, Computers and Mathematics with Applications, 62 (2011) 214 – 224. DOI: 10.1016/j.camwa.2011.04.068
[7] T.G.Ergashev, Generalized Holmgren problem for an elliptic equation with several singular coefficients, Differential Equations, 56(7) (2020) 842–856. DOI: 10.1134/S0012266120070046.
[8] T.G.Ergashev, Z.R.Tulakova, The Dirichlet problem for an elliptic equation with several singular coefficients in an infinite domain, Russian Mathematics, 65(7) (2021) 71–80. DOI: 10.3103/S1066369X21070082
[9] T.G.Ergashev, Z.R.Tulakova, The Neumann problem for a multidimensional elliptic equation with several singular coefficients in an infinite domain. Lobachevskii Journal of Mathematics, 43(1) (2022) 199 – 206. DOI: 10.1134/S19950802220401026

[10] A.S. Berdyshev, A.R.Ryskan. The Neumann and Dirichlet problems for one four-dimensional degenerate elliptic equation. Lobachevskii Journal of Mathematics, 41(6) (2020) 1051–1066. DOI: 10.1134/S1995080220060062
[11] P.Appell, J.Kampe de Feriet, Fonctions Hypergeometriques et Hyperspheriques; Polynomes d'Hermite. Paris: Gauthier – Villars, 1926.
[12] A.Erdelyi, W.Magnus, F.Oberhettinger, F.G.Tricomi, Higher transcendental functions. New York, Toronto, London, McGraw-Hill, 1953.
[13] H.M.Srivastava, P.W.Karlsson, Multiple Gaussian hypergeometric series. New York,Chichester,Brisbane and Toronto: Halsted Press, 1985.
[14] J.Murley, N.Saad, Tables of the Appell hypergeometric functions F_2, ArXiv:0809.5203v2 (2008)
[15] M.O.Abbasova, T.G.Ergashev, Tables of the Lauricella hypergeometric functions $F_A^{(3)}$, Vestnik KRAUNC. Fiz.-Mat. Nauki, 54(1) (2026) 9 – 32. DOI:10.26117/2079-6641-2026-54-1-9-32
[16] A.P.Prudnikov, Yu.A.Brychkov, O.I.Marichev. Integrals and Series. Vol. 3. More Special Functions. Amsterdam, Gordon and Breach Science Publishers, 1990.
[17] J.L.Burchnall, T.W.Chaundy, Expansions of Appell's double hypergeometric functions,The Quarterly J. of Mathematics, Oxford, 11 (1940) 249 – 270.
[18] R.Gilbert, Function Theoretic methods in partial differential equations. New York, London, Academic Press, 1969.
[19] A.V.Bitsadze, Some classes of partial differential equations. Moscow, Nauka, 1981.
[20] M.M.Smirnov, Degenerating elliptic and hyperbolic equations. Moscow, Nauka, 1966.
[21] M.O.Abbasova, T.G.Ergashev, Dirichlet problems for the Laplace equation in infinite domains of Euclidean space, Scientific Bulletin of Namangan State University, 1 (2026) 15 – 20.
[22] M.O.Abbasova, T.G.Ergashev, Neumann problems for Laplace in infinite parts of Euclidean space, Scientific Bulletin of the Bukhara State University, 1 (2026) 44 – 50.

JMMME

JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING



$$\frac{dx}{dt} = Ax + b(t)$$

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u + f$$

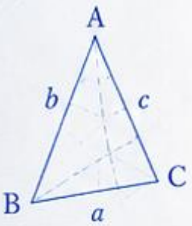
$$\sum_{i=1}^{n} a_i x_i + b$$

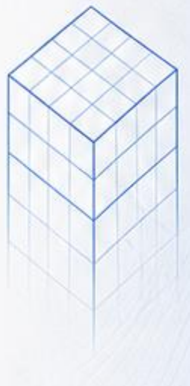
$$f(x) = \int_a^b f(x)\, dx$$

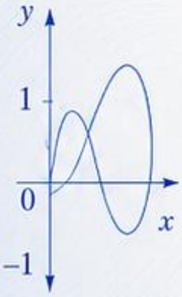
CONTENTS

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\sum_{i=1}^{n} w_i x_i$$







TIIAME National Research University