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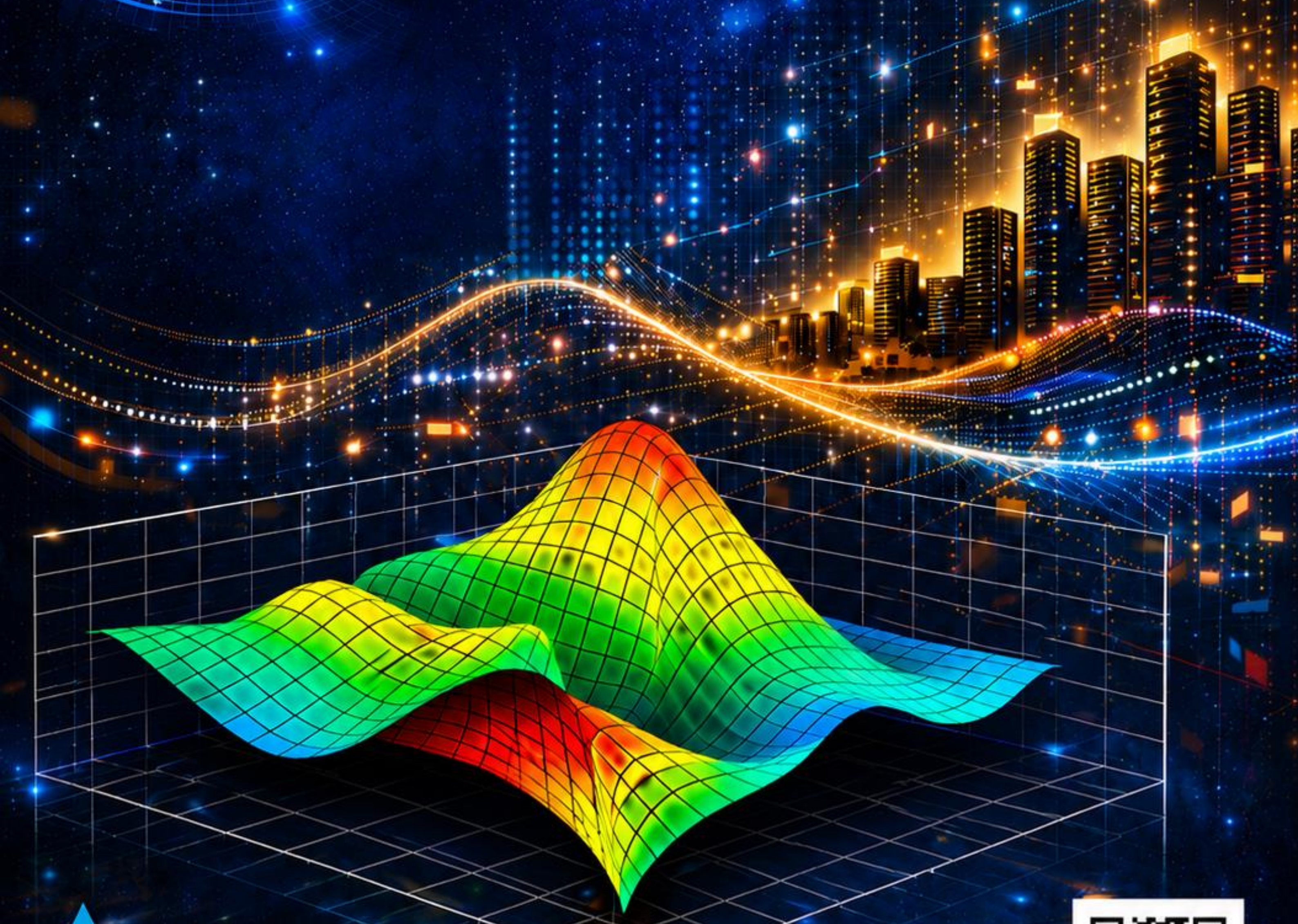
JOURNAL OF MATHEMATICAL MODELS AND METHODS IN ENGINEERING



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"Journal of Mathematical Models and Methods in Engineering" is an international, peer-reviewed, and open-access scientific journal established in the Republic of Uzbekistan by the National Research University "Tashkent Institute of Irrigation and Agricultural Mechanization Engineers" under Certificate No. 1648073 dated June 15, 2026.

The journal specializes in publishing scientific research on mathematical modeling, numerical methods, computational algorithms, optimization, and analytical approaches across various fields of engineering — including civil engineering, mechanical engineering, energy, hydraulic engineering, materials science, robotics, and environmental protection. It serves as a scientific platform for scholars, engineers, educators, and doctoral researchers to exchange theoretical and applied knowledge.

All submitted manuscripts undergo a double-blind peer review process and are published in strict compliance with high academic quality standards and publication ethics.

The journal has been published since June 15, 2026, once per quarter, that is, 4 times per year. Manuscripts are accepted in English and must be prepared in accordance with international IMRAD standards.

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FOUNDER	TIIAME National Research University
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PHONE	+998 (93) 397-12-39
WEBSITE	jmmme.uz
CERTIFICATE	No. 1648073, June 15, 2026
PUBLISHED	2026-06-15
LANGUAGE	English
FREQUENCY	Quarterly / 4 times per year
REVIEW	Double-blind peer review
STANDARD	International IMRAD standards

INDEXED AND DISCOVERABLE THROUGH

ISSN INTERNATIONAL STANDARD SERIAL NUMBER · Google Scholar · zenodo · doi · OpenAIRE · OPEN ACCESS

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Journal of Mathematical Models and Methods in Engineering

journal homepage: https://jmmme.uz/index.php/journal

A three-variable analogue of the Humbert function with applications to solving Dirichlet problem for a singular Helmholtz equation

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ARTICLE INFO

Keywords:
confluent hypergeometric function of three variables; system of partial differential equations; asymptotic formula; three-dimensional Helmholtz equation with three singular coefficients; Dirichlet problem in the first infinite octant

ABSTRACT

In this paper we introduce a new confluent hypergeometric function of three variables, list its elementary properties, and construct a system of partial differential equations that the new function satisfies. We study the behavior of the function and establish an asymptotic formula for a large values of arguments. The results obtained are applied to solving the Dirichlet problem for the three-dimensional Helmholtz equation with three singular coefficients in the first infinite octant. The uniqueness of the solution of the Dirichlet problem in an infinite domain is proved using the extremum principle for elliptic equations. Thanks to the established properties of the confluent hypergeometric function of three variables, the unique solution to the posed boundary value problem is written out in explicit form.

1. Introduction

In study of many applied problems involving thermal conductivity and dynamics, electromagnetic oscillation and aerodynamics, and quantum mechanics and potential theory arise a hypergeometric (higher and special or transcendent) functions [1, 2, 3] which are often referred to as special functions of mathematical physics.

The success of the theory of Gaussian hypergeometric function stimulated the development of the corresponding theory in two or more variables. Much credit for the further development of the theory of hypergeometric functions in two variables goes to Horn [4], who gave a more general definition and introduced the concept of the order of double hypergeometric series. He investigated the convergence of the double hypergeometric functions and established systems of partial differential equations that they satisfy. He divided hypergeometric functions into complete and confluent functions and compiled a list (Horn List) of the second order hypergeometric functions in two variables. There are essentially 34 (14 complete and 20 confluent) convergent functions of the second order.

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Lauricella [5, p. 114] first introduced 14 complete triple hypergeometric series (functions) of the second order and Saran [6] systematically studied of these ten hypergeometric functions in three variables which belong to the Lauricella's set. Other hypergeometric functions in three variables studied in the literature include, in addition to the three-variable analogues of the hypergeometric functions in many variables, the functions introduced by Dhawan [7], Samar [8], and Exton [9]. An expansion of the results on the triple hypergeometric series together with references to the original literature are to be found in the book by Srivastava and Karlsson [10], which is a normal work on the subject. This monograph also contains a full list of all relevant articles up to 1985. For instance, Srivastava and Karlsson [10] presented a table of 205 distinct complete Gaussian functions in three variables together with their sources, if known.

Despite many applications of confluent forms of hypergeometric functions of more than two variables, they have been relatively little studied. In the works of Jain [11] and Exton [12], some functions were studied that are confluent forms of the complete hypergeometric functions in three variables.

When the number of the variables exceeds two or three, some confluent forms of the multiple Lauricella functions are defined in [10, pp. 34 - 36] as an applications of which, the self-similar and fundamental solutions of the second order multidimensional systems of partial differential equations with singular coefficients are constructed in explicit forms (see [13]).

The plan of this paper is as follows. In Section 2 we briefly give some preliminary information, which will be used later. In Section 3 we define new confluent hypergeometric function in three variables, write the system of partial differential equations, which satisfies new confluent function, find behavior and asymptotic expansion formulas. Sections 4 and 5 are devoted to the applications of the confluent hypergeometric functions. In Section 4 we formalize the Dirichlet problem for the Helmholtz equation with the three singular coefficients in the first infinite octant and prove the uniqueness of the solution to the problem. In Section 5 we find the solution of the Dirichlet in explicit form.

2. Preliminaries

The Gaussian hypergeometric function of the complex varible z is defined inside the circle $|z|<1$ as the sum of the hypergeometric series [14] :

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, c\neq 0,-1,-2,... \tag{1}$$

where a, b, c are arbitrary complex numbers which are called the parameters of the hypergeometric function; $(\lambda)_n=\Gamma(\lambda+n)/\Gamma(\lambda), n=0,1,2,...$; $\Gamma(z)$ is a famous Euler's gamma-function.

The following function

$${}_pF_q\left[\begin{matrix}\alpha_1,...,\alpha_p;\\ \beta_1,...,\beta_q;\end{matrix} z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}, \tag{2}$$

is known as the generalized Gaussian function [14], or simply, the generalized hypergeometric function. Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z, the numerator parameters α_1,..., α_p, and the denominator parameters β_1, ..., β_q take on complex values, provided that $\beta_j\neq 0,-1,-2,...$; $j=1,...,q$.

The Bessel function is defined as a special case of the generalized hypergeometric function:

$$J_\alpha(z)=\frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^{\alpha}{}_0F_1\left(\alpha+1;-\frac{z^2}{4}\right)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{z}{2}\right)^{2m+\alpha}.$$

Appel has defined, in 1880, four hypergeometric functions, F_1 to F_4 which are all analogous to Gauss' $F(a,b;c;z)$. For instance [15],

$$F_3\left(a,a',b,b';c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m\left(a'\right)_n(b)_m\left(b'\right)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\max\left\{|x|,|y|\right\}<1. \quad (3)$$

One of the confluent forms of the Appell function (3) is a following Humbert function

$$\Xi_2\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},|x|<1. \quad (4)$$

which satisfies system of partial differential equations

$$\begin{cases} x(1-x)u_{xx}+yu_{xy}+\left[c-\left(a+b+1\right)x\right]u_x-abu=0, \\ yu_{yy}+xu_{xy}+cu_y-u=0, \end{cases} \quad (5)$$

where $u\equiv\Xi_2(a,b;c;x,y)$.

At first we clarify the behavior of $\Xi_2(a,b;c;x,y)$ as $x\to\infty$. It follows from (4) that the representation

$$\Xi_2\left(a,b;c;x,y\right)=\sum_{n=0}^{\infty}\frac{y^n}{(c)_n n!}F(a,b;c+n;x) \quad (6)$$

is true. Substituting an expansion of the Gaussian hypergeometric function of the form [16, p. 183, Eq. (10.13)]

$$F(a,b;c;z)=O\left(z^{-b}\right)+O\left(z^{-b}\right),a-b\neq 0,\pm1,\pm2,...,z\to\infty$$

into (6) we obtain the following leading term of (6) as $x\to\infty$, $|\arg(-x)|<\pi$ [16, p. 809, Eq. (40.42)]:

$$\Xi_2\left(a,b;c;x,y\right)\sim\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\,{}_0F_1(c-a;y)(-x)^{-a}$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\,{}_0F_1(c-b;y)(-x)^{-b},a\neq b, \quad (7)$$

where ${}_0F_1(a;y)$ is the generalized Gaussian function defined in (2).

Now we use the asymptotic expansion (see [17, p. 76])

$${}_0F_1(\nu;-y)=O\left[y^{(1-2\nu)/4}\cos\left(2\sqrt{y}+\frac{(1-2\nu)\pi}{4}\right)\right]\text{as } y\to\infty. \quad (8)$$

Then from (7) we find that

$$\Xi_2\left(a,b;c;x,y\right)=O\left(x^{-a}y^{(1+2a-2c)/4}\right)+O\left(x^{-b}y^{(1+2b-2c)/4}\right),a\neq b,\ x,y\to\infty. \quad (9)$$

The Humbert function $\Xi_2(a,b;c;x,y)$, a PDE-system (1.5), the behavior formula (7) and the asymptotic expansion (9) have important applications: in the work [18] an explicit solution of the Dirichlet problem for the equation

$$u_{xx}+u_{yy}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y-\lambda^2u=0,0<2\alpha,2\beta<1,-\infty<\lambda<+\infty \quad (10)$$

in the infinite first quadrant $(x,y): x\ge 0, y\ge 0$ is written out through a function (4), the satisfaction of which for the equation (10) is established thanks to the system (5), and the formulas (7) and (9) help to show the fulfillment of the conditions of the Dirichlet problem.

3. Confluent hypergeometric function $\Xi_{22}^{(3)}$ in three variables

This section is devoted to the study of the one confluent form of the triple Lauricella function $F_B^{(3)}$. The new function under consideration is analogous to Humbert's confluent hypergeometric function of two variables $\Xi_2(a,b;c;x,y)$ defined in (1.4), and this work is in some degree complementary to the investigation by Exton [12] on the triple confluent hypergeometric functions.

The Lauricella hypergeometric function $F_B^{(3)}$ is defined by the formula [5]

$$F_B^{(3)}\left(a,a',a'',b,b',b'';c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_m\left(a'\right)_n\left(a''\right)_p(b)_m\left(b'\right)_n\left(b''\right)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\quad (11)$$

$$\max\{|x|,|y|,|z|\}<1.$$

The function (11) has 6 confluent forms [12], one of which can be written as follows

$$\Xi_{22}^{(3)}\left(a,b,a',b';c;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a)_m(b)_m\left(a'\right)_n\left(b'\right)_n}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\max\{|x|,|y|\}<1.\quad (12)$$

Note, the multidimensional analogous of the confluent hypergeometric function $\Xi_{22}^{(3)}$ is found in [13].

If one sets

$$\theta=x\frac{\partial}{\partial x},\phi=y\frac{\partial}{\partial y},\psi=z\frac{\partial}{\partial z},\quad (13)$$

it is clear that the new function $\Xi_{22}^{(3)}$ defined in (12) satisfies the following system of PDE [12]:

$$\begin{cases}\left[\theta(\theta+\phi+\psi+c-1)-x(a+\theta)(b+\theta)\right]\Xi_{22}^{(3)}=0,\\ \left[\phi(\theta+\phi+\psi+c-1)-y(a'+\phi)(b'+\phi)\right]\Xi_{22}^{(3)}=0,\\ \left[\psi(\theta+\phi+\psi+c-1)\right]\Xi_{22}^{(3)}=0.\end{cases}$$

Using the operators (13), it is easy to get

$$\begin{cases}x(1-x)u_{xx}+yu_{xy}+zu_{xz}+\left[c-\left(a+b+1\right)x\right]u_x-abu=0,\\ y(1-y)u_{yy}+xu_{xy}+zu_{yz}+\left[c-\left(a'+b'+1\right)y\right]u_y-a'b'u=0,\\ zu_{zz}+xu_{xz}+yu_{yz}+cu_z-u=0,\end{cases}\quad (14)$$

where $u\equiv\Xi_{22}^{(3)}\left(a,b,a',b';c;x,y,z\right)$.

At first we clarify the behavior of $\Xi_{22}^{(3)}\left(a,b,a',b';c;x,y,z\right)$ as $x,y\to\infty$. It follows from (12) that the representation

$$\Xi_{22}^{(3)}\left(a,b,a',b';c;x,y,z\right)=\sum_{n=0}^{\infty}\frac{\left(a'\right)_n\left(b'\right)_n y^n}{n!\left(c\right)_n}\Xi_2\left(a,b;c+n;x,z\right) \qquad (15)$$

is true, where Ξ_2 is the Humbert function defined in (4). Substituting an expansion (9) of the Humbert function Ξ_2 into (15) we obtain the following leading term of (15) as $x,y\to\infty$, $|\arg(-x)|<\pi$, $|\arg(-y)|<\pi$:

$$\Xi_{22}^{(3)}\left(a,b,a',b';c;x,y,z\right)\sim\frac{\Gamma(c)\Gamma(b-a)\Gamma(b'-a')}{\Gamma(b)\Gamma(b')\Gamma(c-a-a')}\,{}_0F_1\left(c-a-a';z\right)(-x)^{-a}(-y)^{-a'}+$$

$$+\frac{\Gamma(c)\Gamma(b-a)\Gamma(a'-b')}{\Gamma(b)\Gamma(a')\Gamma(c-a-b')}\,{}_0F_1\left(c-a-b';z\right)(-x)^{-a}(-y)^{-b'}+$$

$$+\frac{\Gamma(c)\Gamma(a-b)\Gamma(b'-a')}{\Gamma(a)\Gamma(b')\Gamma(c-b-a')}\,{}_0F_1\left(c-b-a';z\right)(-x)^{-b}(-y)^{-a'}+$$

$$+\frac{\Gamma(c)\Gamma(a-b)\Gamma(a'-b')}{\Gamma(a)\Gamma(a')\Gamma(c-b-b')}\,{}_0F_1\left(c-b-b';z\right)(-x)^{-b}(-y)^{-b'},\, a\neq b, a'\neq b'. \qquad (16)$$

Now we use the asymptotic expansion (8). Then from (16) we find that

$$\Xi_{22}^{(3)}\left(a,b,a',b';c;x,y,z\right)=O\left(x^{-a}y^{-a'}z^{(1-2c+2a+2a')/4}\right)$$
$$+O\left(x^{-a}y^{-b'}z^{(1-2c+2a+2b')/4}\right)+O\left(x^{-b}y^{-a'}z^{(1-2c+2b+2a')/4}\right)$$
$$+O\left(x^{-b}y^{-b'}z^{(1-2c+2b+2b')/4}\right), a\neq b, a'\neq b'; x,y,z\to\infty. \qquad (17)$$

4. The Dirichlet problem in the first octant for the Helmholtz equation with three singular coefficients

Let us consider the following singular Helmholtz equation

$$u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z-\lambda^2u=0,\ 0<2\alpha,2\beta,2\gamma<1 \qquad (18)$$

in the infinite domain $\Omega\equiv\{(x,y,z):x>0,y>0,z>0\}$.

The Dirichlet problem. Find a regular solution $u(x,y,z)$ to the singular Helmholtz equation (18) which belongs to the class of functions $C\left(\overline{\Omega}\right)\cup C^2(\Omega)$, satisfying the conditions

$$u(x,y,0)=\tau_1(x,y),\ 0\le x,y<\infty, \qquad (19)$$
$$u(x,0,z)=\tau_2(x,z),\ 0\le x,z<\infty, \qquad (20)$$
$$u(0,y,z)=\tau_3(y,z),\ 0\le y,z<\infty, \qquad (21)$$
$$\lim_{R\to\infty}u(x,y,z)=0, R=\sqrt{x^2+y^2+z^2}, \qquad (22)$$

where $\tau_k(t,s)$ are given functions such that

$$\tau_1(x,y)=\frac{\tilde{\tau}_1(x,y)}{\left(1+x^2+y^2\right)^{\left(1+\alpha+\beta-\gamma+\varepsilon_1\right)/2}},\tilde{\tau}_1(x,y)\in C(0\le x,y<\infty),\varepsilon_1>2-2\alpha-2\beta, \qquad (23)$$

$$\tau_2(x,z)=\frac{\tilde{\tau}_2(x,z)}{\left(1+x^2+z^2\right)^{\left(1+\alpha-\beta+\gamma+\varepsilon_2\right)/2}},\tilde{\tau}_2(x,z)\in C(0\le x,z<\infty),\varepsilon_2>2-2\alpha-2\gamma, \qquad (24)$$

$$\tau_3(y,z)=\frac{\tilde{\tau}_3(y,z)}{\left(1+y^2+z^2\right)^{\left(1-\alpha+\beta+\gamma+\varepsilon_3\right)/2}},\tilde{\tau}_3(y,z)\in C(0\le y,z<\infty),\varepsilon_3>2-2\beta-2\gamma. \qquad (25)$$

In addition, the functions $\tau_1(x,y)$, $\tau_2(x,z)$ and $\tau_3(y,z)$ satisfy the concordance conditions in origin $\tau_1(0,0)=\tau_2(0,0)=\tau_3(0,0)$, and in the lateral ribs of the domain Ω:

$$\tau_1(x,0)=\tau_2(x,0),\tau_1(0,y)=\tau_3(y,0),\tau_2(0,z)=\tau_3(0,z)\,0\le x,y,z<\infty. \qquad (26)$$

In the two-dimensional case the Dirichlet problem for the Helmholtz equation (10) in the infinite first quadrant is studied in [18].

In the paper [19] the Dirichlet problem for an elliptic equation with two singular coefficients

$$u_{xx}+u_{yy}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y=0,0<2\alpha,2\beta<1$$

in the arbitrary domain bounded in the first quarter $\{(x,y):x>0,y>0\}$ of the xOy-plane is solved by a potential theory method and in the paper [20] a solution of the Dirichlet problem for multidimensional elliptic equation with two singular coefficients

$$\sum_{j=1}^{m}u_{x_jx_j}+\frac{2\alpha}{x_1}u_{x_j}+\frac{2\beta}{x_2}u_{x_2}=0,0<2\alpha,2\beta<1$$

in the quarter of the multidimensional ball is found in the explicit form. In the work [21] a spatial mixed problems and Neumann problem for a three- dimensional elliptic equation with two singular coefficients

$$u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y=0,0<2\alpha,2\beta<1$$

are investigated.

Conditions (23) – (25) are essential for the unique solvability of the Dirichlet problem. A similar problem for the equation (18) under more severe constraints on the data is solved in [22].

Theorem 1. *The Dirichlet problem for equation (18) in an unbounded domain* Ω *can have at most one solution.*

Proof. For the proof of the theorem, it is sufficient to show that the corresponding homogeneous problem has a trivial solution. For that we denote by Ω_R the finite part of domain Ω, which is bounded by planes $x=0$, $y=0$, $z=0$ and a part of the sphere

$$\sigma_R:=\{(x,y,z):x^2+y^2+z^2,x>0,y>0,z>0\}.$$

Let

$$\tau_1(x,y)=0,\tau_2(x,z)=0,\tau_3(y,z)=0. \qquad (27)$$

The validity of theorem follows from the extremum principle for the elliptic equations [23]. Indeed, in the domain $\overline{\Omega}_R$ due to (10), the function $u(x,y,z)$ can attain its positive maximum and negative minimum on σ_R only.

Let (x,y,z) be an arbitrary point of domain Ω_R. Let us take an arbitrary small number $\varepsilon>0$ and, taking into account (10), we choose R so large that $|u(x,y,z)|<\varepsilon$ on σ_R. With suffuciently large R, this point belongs to the domain Ω_R and therefore $|u(x,y,z)|<\varepsilon$. In view of arbitrariness of ε we have $u(x,y,z)=0$. Then $u(x,y,z)=0$ in the domain Ω.

5. Existence of the solution to the Dirichlet problem

We will seek a solution to the Dirichlet problem (18) – (26) in the form

$$u(x,y,z)=\sum_{i=1}^{3}u_i(x,y,z),$$

where $u_1(x,y,z)$, $u_2(x,y,z)$, $u_3(x,y,z)$ are a solutions to the equation (18), satisfying the boundary conditions

$$u_1(x,y,0)=\tau_1(x,y), u_1(x,0,z)=0, u_1(0,y,z)=0, \tag{28}$$

$$u_2(x,y,0)=0, u_2(x,0,z)=\tau_2(x,z), u_2(0,y,z)=0, \tag{29}$$

$$u_3(x,y,0)=0, u_3(x,0,z)=0, u_3(0,y,z)=\tau_3(y,z), \tag{30}$$

respectively.

Lemma 1. *If a function* $\tau_1(x,y)$ *can be represented by the formula (23), then the function*

$$u_1(x,y,z)=\sum_{j=1}^{4}(-1)^{j+1}u_{1j}(x,y,z) \tag{31}$$

is a regular solution of the equation (18) in the domain Ω, satisfying the boundary conditions(28) and vanishing at infinity, where

$$u_{1j}(x,y,z)=\frac{2(1-2\gamma)x^{-\alpha}y^{-\beta}z^{1-2\gamma}}{\pi}\int_0^\infty\int_0^\infty\frac{\tau_1(t,s)t^{\alpha}s^{\beta}}{\rho_{1j}^{3-2\gamma}}\times$$

$$\times\Xi_{22}^{(3)}\left(\alpha,1-\alpha,\beta,1-\beta;\gamma-\frac{1}{2};\frac{\rho_{1j}^2}{-4xt};\frac{\rho_{1j}^2}{-4ys};\frac{\lambda^2}{4}\rho_{1j}^2\right)dtds,\ j=1,2,3,4; \tag{32}$$

$$\rho_{11}=\sqrt{(t-x)^2+(s-y)^2+z^2},\rho_{12}=\sqrt{(t+x)^2+(s-y)^2+z^2},$$

$$\rho_{13}=\sqrt{(t+x)^2+(s+y)^2+z^2},\rho_{14}=\sqrt{(t-x)^2+(s+y)^2+z^2},$$

and $\Xi_{22}^{(3)}$ is confluent hypergeometric function in three variables defined in (12).

Proof. First, we need to make sure that function (32) satisfies the equation

$$W_{xx}+W_{yy}+W_{zz}+\frac{2\alpha}{x}W_x+\frac{2\beta}{y}W_y+\frac{2\gamma}{z}W_z-\lambda^2W=0 \tag{33}$$

in the arguments x, y and z. For this purpose, consider the function

$$W(x,y,z)=x^{-\alpha}y^{-\beta}z^{1-2\gamma}\rho^{\gamma-\frac{3}{2}}\omega(\xi,\eta,\zeta),$$

where

$$\omega(\xi,\eta,\zeta)=\Xi_{22}^{(3)}\left(\alpha,1-\alpha,\beta,1-\beta;\gamma-\frac{1}{2};\frac{\rho}{-4xt};\frac{\rho}{-4ys};\frac{\lambda^2}{4}\rho\right), \tag{34}$$

$$\rho=(x-t)^2+(y-s)^2+z^2,\xi=-\frac{\rho}{4xt},\eta=-\frac{\rho}{4ys},\zeta=\frac{\lambda^2}{4}\rho.$$

Calculating the necessary derivatives of function W with respect to variables x, y, z and substituting them into equation (33), we obtain a following system of partial differential equations

$$\begin{cases} \xi(1-\xi)\omega_{\xi\xi}+\eta\omega_{\xi\eta}+\zeta\omega_{\xi\zeta}+\left(\gamma-\dfrac{1}{2}-2\xi\right)\omega_{\xi}-\alpha(1-\alpha)\omega=0, \\ \eta(1-\eta)\omega_{\eta\eta}+\xi u_{\xi\eta}+\zeta\omega_{\eta\zeta}+\left(\gamma-\dfrac{1}{2}-2\eta\right)\omega_{\eta}-\beta(1-\beta)\omega=0, \\ \zeta\omega_{\zeta\zeta}+\xi\omega_{\xi\zeta}+\eta\omega_{\eta\zeta}+\left(\gamma-\dfrac{1}{2}\right)\omega_{\zeta}-\omega=0. \end{cases} \tag{35}$$

Comparing the system of equations (35) with the system of hypergeometric type equations (14), we can conclude that the function $\omega(\xi,\eta,\zeta)$ defined by formula (34), is a solution to system of equations (35). Thus, function (32) satisfies equation (18).

Now we will show that function $u_{11}(x,y,z)$ satisfies condition (19) of the Dirichlet problem.

By setting $t=x+z\mu, s=y+z\nu,$ we transform formula (32) for $j=1$ to the form

$$u_{11}(x,y,z)=\frac{2(1-2\gamma)}{\pi}\int_0^\infty\int_0^\infty\frac{\tau_1(x+z\mu,y+z\nu)}{w^{(3-2\gamma)/2}}\left(\frac{x+z\mu}{x}\right)^{\alpha}\left(\frac{y+z\nu}{y}\right)^{\beta}\times$$

$$\times\Xi_{22}^{(3)}\left(\alpha,1-\alpha,\beta,1-\beta;\gamma-\frac{1}{2};\frac{wz^2}{-4x(x+z\mu)};\frac{wz^2}{-4y(y+\nu)};\frac{\lambda^2}{4}wz^2\right)d\mu d\nu, \tag{36}$$

where $w:=1+\mu^2+\nu^2$ for brevity.

On the right-hand side of (36) we pass to the limit as $z\to 0$ and get

$$\lim_{z\to 0}u_{11}(x,y,z)=\tau_1(x,y)\frac{2(1-2\gamma)}{\pi}\int_0^\infty\int_0^\infty\frac{d\mu d\nu}{\left(1+\mu^2+\nu^2\right)^{(3-2\gamma)/2}}.$$

Using consistently the well-known identity [24, p. 612, Eq. 4.623]

$$\int_0^\infty\int_0^\infty\varphi\left(a^2x^2+b^2y^2\right)dxdy=\frac{\pi}{2ab}\int_0^\infty\varphi\left(x^2\right)xdx, \tag{37}$$

we obtain

$$\lim_{z\to 0}u_{11}(x,y,z)=\tau_1(x,y).$$

Likewise, everyone can easily get

$$\lim_{z\to 0}u_{1j}(x,y,z)=0,\ j=2,3,4.$$

So, the following equality is true for the function $u_1(x,y,z)$ defined in (31):

$$\lim_{z\to 0}u_1(x,y,z)=\tau_1(x,y).$$

Applying the behavior formula (16) to (32) and taking into account the obvious equalities

$$\rho_{11}\big|_{x=0}=\rho_{12}\big|_{x=0},\rho_{11}\big|_{y=0}=\rho_{14}\big|_{y=0},\rho_{12}\big|_{y=0}=\rho_{13}\big|_{y=0},\rho_{13}\big|_{x=0}=\rho_{14}\big|_{x=0},$$

we establish that for $\alpha<\dfrac{1}{2}$ and $\beta<\dfrac{1}{2}$ there exist a following limits

$$\lim_{x\to 0}u_1(x,y,z)=0,\ \lim_{y\to 0}u_1(x,y,z)=0.$$

It is easy to see that as the point (x,y,z) tends to infinity, i.e. $R\to\infty$, function (32) tends to zero. To do this, we will use the formula (17) that describes the behavior of the function $\Xi_{22}^{(3)}$ at infinity. Then, by virtue of (23), we get

$$\left|u_{11}\left(x,y,z\right)\right|\le C_{11}z^{1-2\gamma}\int\limits_0^\infty\int\limits_0^\infty\frac{t^{2\alpha}s^{2\beta}dtds}{\left(1+t^2+s^2\right)^{\left(1+\alpha+\beta-\gamma+\varepsilon_1\right)/2}\rho_{11}^{2+\alpha+\beta-\gamma}}$$

$$+C_{12}y^{1-2\beta}z^{1-2\gamma}\int\limits_0^\infty\int\limits_0^\infty\frac{t^{2\alpha}sdtds}{\left(1+t^2+s^2\right)^{\left(1+\alpha+\beta-\gamma+\varepsilon_1\right)/2}\rho_{11}^{3+\alpha-\beta-\gamma}}$$

$$+C_{13}x^{1-2\alpha}z^{1-2\gamma}\int\limits_0^\infty\int\limits_0^\infty\frac{ts^{2\beta}dtds}{\left(1+t^2+s^2\right)^{\left(1+\alpha+\beta-\gamma+\varepsilon_1\right)/2}\rho_{11}^{3+\beta-\gamma-\alpha}}$$

$$+C_{14}x^{1-2\alpha}y^{1-2\beta}z^{1-2\gamma}\int\limits_0^\infty\int\limits_0^\infty\frac{tsdtds}{\left(1+t^2+s^2\right)^{\left(1+\alpha+\beta-\gamma+\varepsilon_1\right)/2}\rho_{11}^{4-\alpha-\beta-\gamma}}. \tag{38}$$

At the right-hand sides of inequalities (38), change the variables as follows: $t=R\mu$, $s=R\nu$. We obtain the inequalities

$$\left|u_{11}\right|\le\frac{1}{R^{2\alpha+2\beta-2+\varepsilon_1}}\left(\frac{z}{R}\right)^{1-2\gamma}\left[\frac{C_{11}}{R^{2-2\alpha-2\beta}}K_1(x,y,z;R)+\frac{C_{12}}{R^{1-2\alpha}}\left(\frac{y}{R}\right)^{1-2\beta}K_2(x,y,z;R)\right.$$

$$\left.+\frac{C_{13}}{R^{1-2\beta}}\left(\frac{x}{R}\right)^{1-2\alpha}K_3(x,y,z;R)+C_{14}\left(\frac{x}{R}\right)^{1-2\alpha}\left(\frac{y}{R}\right)^{1-2\beta}K_4(x,y,z;R)\right],$$

where

$$K_1(x,y,z;R)=\int\limits_0^\infty\int\limits_0^\infty\frac{\mu^{2\alpha}\nu^{2\beta}d\mu d\nu}{\Lambda^{\left(1+\alpha+\beta-\gamma+\varepsilon_1\right)/2}\Upsilon^{(2+\alpha+\beta-\gamma)/2}},$$

$$K_2(x,y,z;R)=\int\limits_0^\infty\int\limits_0^\infty\frac{\mu^{2\alpha}\nu d\mu d\nu}{\Lambda^{\left(1+\alpha+\beta-\gamma+\varepsilon_1\right)/2}\Upsilon^{(3+\alpha-\beta-\gamma)/2}},$$

$$K_3(x,y,z;R)=\int\limits_0^\infty\int\limits_0^\infty\frac{\mu\nu^{2\beta}d\mu d\nu}{\Lambda^{\left(1+\alpha+\beta-\gamma+\varepsilon_1\right)/2}\Upsilon^{(3+\beta-\gamma-\alpha)/2}},$$

$$K_4(x,y,z;R)=\int\limits_0^\infty\int\limits_0^\infty\frac{\mu\nu d\mu d\nu}{\Lambda^{\left(1+\alpha+\beta-\gamma+\varepsilon_1\right)/2}\Upsilon^{(4-\beta-\gamma-\alpha)/2}},$$

$$\Lambda=\frac{1}{R^2}+\mu^2+\nu^2,\Upsilon=1+\mu^2+\nu^2-\frac{2}{R}\left(\mu x+\nu y\right).$$

Let us show that these proper double integrals are bounded. Indeed, using the formula (37) and passing to the limit as $R\to\infty$, we find the relations

$$\lim_{R\to\infty}R_j\left(x,y,z;R\right)=M_j,\ j=1,2,3,4, \tag{39}$$

where

$$M_1=\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+2\gamma+\varepsilon_1}{2}\right)\Gamma\left(\frac{2-2\gamma+\varepsilon_1}{2}\right)}{4\Gamma(1+\alpha+\beta)\Gamma\left(\frac{2+\alpha+\beta-\gamma}{2}\right)},$$

$$M_2=\frac{\Gamma\left(\frac{2+\alpha-\beta+\gamma-\varepsilon_1}{2}\right)\Gamma\left(\frac{1-2\gamma+\varepsilon_1}{2}\right)}{2(1+2\alpha)\Gamma\left(\frac{3+\alpha-\beta-\gamma}{2}\right)},$$

$$M_3=\frac{\Gamma\left(\frac{2-\alpha+\beta+\gamma-\varepsilon_1}{2}\right)\Gamma\left(\frac{1-2\gamma+\varepsilon_1}{2}\right)}{2(1+2\beta)\Gamma\left(\frac{3-\alpha+\beta-\gamma}{2}\right)},$$

$$M_4=\frac{\Gamma\left(\frac{3-\alpha-\beta+\gamma-\varepsilon_1}{2}\right)\Gamma\left(\frac{1-2\gamma+\varepsilon_1}{2}\right)}{4\Gamma\left(\frac{4-\alpha-\beta-\gamma}{2}\right)}.$$

From (39), it intermediately follows that

$$\left|u_{11}(x,y,z)\right|\le\frac{C}{R^{2\alpha+2\beta-2+\varepsilon_1}},\varepsilon_1>2-2\alpha-2\beta, \tag{40}$$

where C is a constant.

Now, it intermediately follows from estimate (40) that the function (32) vanishes at infinity. The **Lemma 1** is proved.

Similar statements are also true for functions $u_2(x,y,z)$ and $u_3(x,y,z)$ that satisfy the conditions (29) and (30), respectively, and vanish at infinity, where

$$u_i(x,y,z)=\sum_{j=1}^{4}(-1)^{j+1}u_{ij}(x,y,z),i=2,3,$$

$$u_{2j}(x,y,z)=\frac{2(1-2\beta)}{\pi}x^{-\alpha}y^{1-2\beta}z^{-\gamma}\int_0^\infty\int_0^\infty\frac{\tau_2(t,s)t^\alpha s^\gamma}{\rho_{2j}^{3-2\beta}}\times$$
$$\times\Xi_{22}^{(3)}\left(\alpha,1-\alpha,\gamma,1-\gamma;\beta-\frac{1}{2};\frac{\rho_{2j}^2}{-4xt};\frac{\rho_{2j}^2}{-4zs};\frac{\lambda^2}{4}\rho_{2j}^2\right)dtds, \tag{41}$$

$$\rho_{21}=\sqrt{(t-x)^2+y^2+(s-z)^2},\ \rho_{22}=\sqrt{(t+x)^2+y^2+(s-z)^2},$$

$$\rho_{24}=\sqrt{(t-x)^2+y^2+(s+z)^2},\ \rho_{23}=\sqrt{(t+x)^2+y^2+(s+z)^2},$$

$$u_{3j}(x,y,z)=\frac{2(1-2\alpha)}{\pi}x^{1-2\alpha}y^{-\beta}z^{-\gamma}\int_0^\infty\int_0^\infty\frac{\tau_3(t,s)t^\beta s^\gamma}{\rho_{3j}^{3-2\alpha}}\times$$

$$\times \Xi_{22}^{(3)}\left(\beta,1-\beta,\gamma,1-\gamma;\alpha-\frac{1}{2};\frac{\rho_{3j}^2}{-4yt};\frac{\rho_{3j}^2}{-4zs};\frac{\lambda^2}{4}\rho_{3j}^2\right)dtds, \qquad (42)$$

$$\rho_{31}=\sqrt{x^2+(t-y)^2+(s-z)^2},\ \rho_{32}=\sqrt{x^2+(t+y)^2+(s-z)^2},$$

$$\rho_{34}=\sqrt{x^2+(t-y)^2+(s+z)^2},\ \rho_{33}=\sqrt{x^2+(t+y)^2+(s+z)^2}.$$

Thus, we have proved the following theorem

Theorem 2. *Function* $u(x,y,z)=\sum_{i=1}^{3}\sum_{j=1}^{4}(-1)^{j+1}u_{ij}(x,y,z)$, *where the functions* u_{ij} *are defined in (32), (41) and (42), is a regular solution of equation (18) in the domain* Ω, *satisfying the boundary conditions (19) to (22).*

6. Conclusion

In this paper we dealt with the three-dimensional singular Helmholtz equation of the second order. In the future, the study can be developed in two directions: firstly, towards increasing the dimensionality of the equation, secondly, towards increasing the order of the equation, it would be necessary to consider the above-studied problem at least for the third order singular partial differential equation.

Because, at the present all fundamental solutions of the multidimensional Helmholtz equation with the singular coefficents

$$\sum_{i=1}^{m}\frac{\partial^2 u}{\partial x_i^2}+\sum_{k=1}^{n}\frac{2\alpha_i}{x_k}\frac{\partial u}{\partial x_k}-\lambda^2 u=0, 0<2\alpha_k<1, m\geq 2, 1\leq n\leq m \qquad (43)$$

are expressed by a multiple confluent hypergeometric function defined in [25]. In addition, particular solutions of the more general equation

$$\sum_{j=1}^{k}\frac{\partial^2 u}{\partial x_j^2}+\sum_{j=1}^{k}\frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j}=\sum_{j=k+1}^{n}\frac{\partial^2 u}{\partial x_j^2}+\sum_{j=k+1}^{n}\frac{2\alpha_j}{x_j}\frac{\partial u}{\partial x_j}, k=\overline{1,n-1}$$

are currently known [26].

In the multidimensional case, only Kapilevič [27], in 1952, has solved the Dirichlet problem for the Helmholtz equation with the single singular coefficient (i.e. $n=1$ in Eq. (43)) in the half-space.

To study third-order partial differential equations, all results on the high-order hypergeometric functions are required. For instance, the following degenerating model parabolic equation of the third order

$$x^n y^m u_t - t^k y^m u_{xxx} - t^k x^n u_{yyy} = 0, m, n, k = const > 0$$

has 9 linearly independent particular solutions [28] which are expressed by, so called, a Kampé de Fériet's hypergeometric functions in two variables [10, p. 27] the order of which exceeds two. For the study of high-order hypergeometric functions, expansion formulas are very important, which allow us to express a given multiple hypergeometric function by the infinite sum of hypergeometric functions of lower order. In a recent work [29], expansions of the Kampé de Fériet's functions are established.

The study can be started with solving the Dirichlet problem for a four-dimensional equation

$$u_{xx}+u_{yy}+u_{zz}+u_{tt}+\frac{2\alpha}{x}u_x+\frac{2\beta}{y}u_y+\frac{2\gamma}{z}u_z+\frac{2\delta}{t}u_t-\lambda^2 u=0,\ 0<\alpha,\beta,\gamma,\delta<1/2 \qquad (44)$$

using the results of the work [30] devoted to the Dirichlet problem for equation (44) at $\lambda=0$.

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$$\frac{dx}{dt} = Ax + b(t)$$

$$\frac{\partial u}{\partial t} = \alpha\nabla^2 u + f$$

$$\sum_{i=1}^{n} a_i x_i + b$$

$$f(x) = \int_a^b f(x)\, dx$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\sum_{i=1}^{n} w_i x_i$$

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